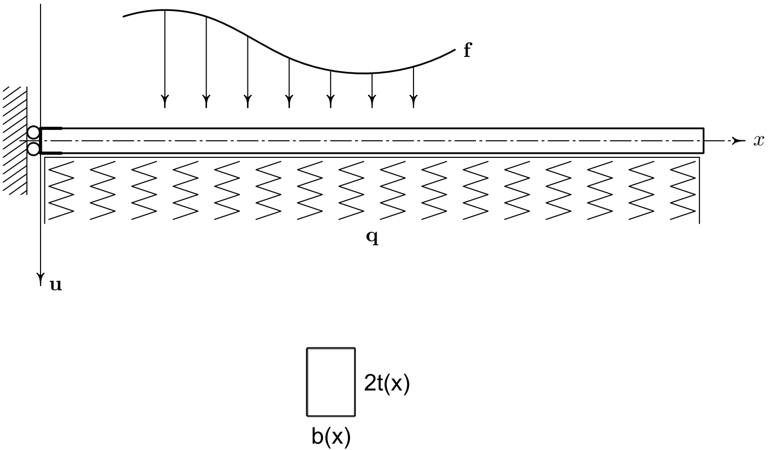
Sizing Optimization of a Beam on an Elastic Unilateral Subsoil

R. Šimeček[1]

[1]Faculty of Science,
Department of Mathematical Analysis and Applications of Mathematics,
Palacky University in Olomouc, Czech Republic

Applied Mathematics Seminar, University of Turku, 7th May 2010

Bending of an elastic beam



Boundary value problem

Find a function $u(x) \in C^4(\Omega) \cap C^3(\bar{\Omega})$ such that

$$\begin{cases} \left(\beta(x)t^3(x)u''(x)\right)'' + q(x)u^+(x) = f(x) & \forall x \in \Omega \\ u'(0) = u''(0) = u''(l) = u'''(0) = 0, \end{cases}$$

where t, q a f are functions representing the thickness of the beam, stifness coefficient of the subsoil and the intensity of load. Function u is the deflection of the beam and u^+ is its positive part

$$u^+(x) = \frac{u(x) + |u(x)|}{2}, \; x \in \Omega.$$

Function β has the following form

$$\beta(x) = \frac{8}{12} b(x)E(x),$$

where E is Young's modulus of elasticity and b is a function corresponding to the width of the beam.

Set of admissible design variables U_{ad}

Subject of optimization is the thickness $t(x)$ and stiffness $q(x)$. The design variables appear in coefficients of the corresponding differential operator, while the domain of integration remain fixed.

Thickness of the beam

$$U_{ad}^{t} = \Big\{ t \in C(\Omega) \quad : \quad 0 < T_0 \leq t(x) \leq T_1 \ \ \forall x \in \Omega, \quad \int_{\Omega} t(x)\,\mathrm{d}x = T_2, \ \ |t'(x)| \leq T_3 \ \ \forall x \in \Omega \Big\}.$$

Subsoil stiffness

$$U_{ad}^{q} = \Big\{ q \in L^2(\Omega) \ : \ Q_0 \leq q(x) \leq Q_1 \ \ \forall x \in \Omega \Big\}.$$

Variational formulation

Let **e** is an arbitrary but fixed pair $[t, q] \in U_{ad}$, $\beta \in L^\infty(\Omega)$ and there exists a constant β_0 such that $0 < \beta_0 \le \beta(x)$ s.v. v Ω.

$$\begin{cases} \text{Find } u^* \in \mathbb{V} \text{ such that} \\ E_{\mathbf{e}}(u^*) \le E_{\mathbf{e}}(v) \;\; \forall v \in \mathbb{V}, \end{cases} \qquad (P(\mathbf{e}))$$

where $E_{\mathbf{e}}$ is the functional of total potential energy of the beam, given by

$$\begin{aligned} E_{\mathbf{e}}(v) &:= \frac{1}{2}\left(a_{\mathbf{e}}(v,v) + b_{\mathbf{e}}(v^+, v^+)\right) - F(v), \\ a_{\mathbf{e}}(u,v) &:= \int_\Omega \beta(x) t^3(x) u''(x) v''(x)\, \mathrm{d}x, \\ b_{\mathbf{e}}(u,v) &:= \int_\Omega q(x) u(x) v(x)\, \mathrm{d}x, \\ F(v) &:= \int_\Omega f(x) v(x)\, \mathrm{d}x \\ \mathbb{V} &:= \{v \in H^2(\Omega) : v'(0) = 0\}. \end{aligned}$$

Existence of solution - state problem

Functional $E_{\mathbf{e}}$ is convex a G-differentiable, but only semicoercive on $H^2(\Omega)$. It is necessary to make a decomposition of $\mathbb{V}$ and prove the coercivity on its subspaces using a modified Poincare inequality.

Necessary and sufficient condition

Let $\beta \in L^\infty(\Omega)$, $\mathbf{e} = [t, q] \in U_{ad}$. The state $(P(\mathbf{e}))$ has unique solution if and only if

$$F(1) > 0$$

Cost functional

Cost functional

Generally the cost functional is defined as a mapping

$$\mathcal{J} : U_{ad} \times \mathbb{V} \to \mathbb{R}.$$

$$\mathcal{J}_1(\mathbf{e}, u(\mathbf{e})) = \int_\Omega f(x)u(x)\,\mathrm{d}x, \quad \mathcal{J}_2(\mathbf{e}, u(\mathbf{e})) = \|u(x)\|^2_{2,2}\,\mathrm{d}x,$$

$$\mathcal{J}_3(\mathbf{e}, u(\mathbf{e})) = \int_\Omega u^2(x)\,\mathrm{d}x, \quad \mathcal{J}_4(\mathbf{e}, u(\mathbf{e})) = \int_\Omega \mathbf{e}^2(u''(x))^2(x)\,\mathrm{d}x.$$

Optimization problem

Optimization problem

$$\begin{cases} \text{Find } \mathbf{e}^* \in U_{ad} \text{ such that} \\ \mathcal{J}(\mathbf{e}^*, u(\mathbf{e}^*)) \leq \mathcal{J}(\mathbf{e}, u(\mathbf{e})) \quad \forall \mathbf{e} \in U_{ad}, \end{cases} \tag{$\mathcal{P}$}$$

where $u(\mathbf{e}) \in \mathbb{V}$ is a solution of the state problem $(P(\mathbf{e}))$.

Outline of the optimization problem

$$\mathbf{e} \longmapsto u(\mathbf{e}) \longmapsto \mathcal{J}(\mathbf{e}, u(\mathbf{e})).$$

Existence of solution

Theorem (Continuous dependence)

Let $\mathbf{e}_n, \mathbf{e} \in U_{ad},\ \mathbf{e}_n \to \mathbf{e}$. Let $u_n(\mathbf{e}_n) \in \mathbb{V}$ is solution to state problem $(P(\mathbf{e}_n))$ and the condition $L(1) > 0$ holds. Then there exists a function $u \in \mathbb{V}$ such that

$$u_n \to u \ \text{ in } \mathbb{V}$$

moreover $u = u(\mathbf{e})$ is a solution to the state problem $(P(\mathbf{e}))$.

Theorem

Let the condition $F(1) > 0$ holds, then there exists at least one solution to the optimization problem $(\mathcal{P})$.

Design variable approximation

Let us define an equidistant partition Δ_κ of interval $\Omega \equiv (0, l)$ with step $\kappa > 0,\ l = n(\kappa)\kappa,\ b_i = i\kappa,\ \forall i = 0, 1, \dots, n(\kappa)$:

$$\Delta_\kappa : 0 = b_0 < b_1 < \cdots < b_{n(\kappa)-1} < b_{n(\kappa)} = l.$$

Instead of general thickness $t(x) \in U^t_{ad}$ we will consider only those functions from U^t_{ad} that are continuous and piecewise linear on Δ_κ. As well as we will consider only those functions from U^q_{ad} that are piecewise constant on Δ_κ

$$\begin{aligned} U^t_{ad,\kappa} &= \left\{ t_\kappa \in C(\Omega) :\ t_\kappa|_{[b_{i-1},b_i]} \in P_1([b_{i-1}, b_i]),\ \forall i = 1, ..., n(\kappa) \right\} \cap U^t_{ad}, \\ U^q_{ad,\kappa} &= \left\{ q_\kappa \in L^\infty(\Omega) :\ q_\kappa|_{[b_{i-1},b_i]} \in P_0([b_{i-1}, b_i]),\ \forall i = 1, ..., n(\kappa) \right\} \cap U^q_{ad}. \end{aligned}$$

The approximation of U_{ad} is defined as $U_{ad,\kappa} = U^t_{ad,\kappa} \times U^q_{ad,\kappa}$.

State problem approximation

We use a finite element approach. Let us define a new equidistant partition Δ_h of interval $\Omega \equiv (0, l)$ with step $h > 0$, $l = n(h)\kappa$, $a_i = ih$, $\forall i = 0, 1, \ldots, n(h)$:

$$\Delta_h : 0 = a_0 < a_1 < \cdots < a_{n(\kappa)-1} < a_{n(h)} = l.$$

Space $\mathbb{V}_h \subset \mathbb{V}$ contains all piecewise cubic polynomials on Δ_h that are continuous together with their first derivatives in Ω. These polynomials satisfy the same stable boundary conditions as functions from $\mathbb{V}$.

$$\mathbb{V}_h = \left\{ v_h \in C^1(\Omega) : \; v_h|_{[a_{i-1}, a_i]} \in P_3([a_{i-1}, a_i]), \; \forall i = 1, \ldots, n(h), \; v_h'(0) = 0 \right\}.$$

State problem approximation

Approximation of the state problem for fixed $\mathbf{e}_\kappa \in U_{ad,\kappa}$

$$\left\{ \begin{array}{l} \text{Find } u_h^* \in \mathbb{V}_h \text{ such that} \\ E^h_{\mathbf{e}_\kappa}(u_h^*) \le E^h_{\mathbf{e}_\kappa}(v_h) \;\; \forall v_h \in \mathbb{V}_h, \end{array} \right. \qquad (P_h(\mathbf{e}_\kappa))$$

where $E^h_{\mathbf{e}_\kappa}$ is the approximated functional of total potential energy of the beam, given by

$$\begin{aligned}
E^h_{\mathbf{e}_\kappa}(v_h) &:= \frac{1}{2}\left(a^h_{\mathbf{e}_\kappa}(v_h, v_h) + b^h_{\mathbf{e}_\kappa}(v_h^+, v_h^+)\right) - F^h(v_h). \\
a^h_{\mathbf{e}}(u, v) &:= \beta \sum_{i=1}^{n(h)} \left(\sum_{j=1}^{m} \omega_{j,i} t^3(z_{j,i}) u''(z_{j,i}) v''(z_{j,i}) \right), \\
b^h_{\mathbf{e}}(u, v) &:= \sum_{i=1}^{n(h)} \left(q_i \sum_{j=1}^{m} \omega_{j,i} u(z_{j,i}) v(z_{j,i}) \right), \\
F^h(v) &:= \sum_{i=1}^{n(h)} \left(\sum_{j=1}^{m} \omega_{j,i} f(z_{j,i}) v(z_{j,i}) \right).
\end{aligned}$$

Approximate optimization problem

The cost functional will be approximated also using the numerical quadrature formula.

$$\mathcal{J}^h(\mathbf{e}, u(\mathbf{e})) := \sum_{i=1}^{n(h)} \left(\sum_{j=1}^{m} \omega_{j,i} \mu(z_{j,i}) \right),$$

where $\mu(x) = u^2(x)$ for example. It depends on the cost functional. The approximated optimization problem is defined in the following way

$$\left\{ \begin{array}{l} \text{Find } \mathbf{e}_\kappa^* \in U_{ad,\kappa} \text{ such that} \\ \mathcal{J}^h(\mathbf{e}_\kappa^*, u_h(\mathbf{e}_\kappa^*)) \le \mathcal{J}^h(\mathbf{e}_\kappa, u_h(\mathbf{e}_\kappa)) \;\; \forall \mathbf{e}_\kappa \in U_{ad,\kappa}, \end{array} \right. \qquad (\mathcal{P}_\kappa)$$

where $u_h(\mathbf{e}_\kappa) \in \mathbb{V}_h$ is a solution to the state problem $(P_h(\mathbf{e}_\kappa))$.

Convergence

State problem

The state $(P^h(\mathbf{e}_\kappa))$ has unique solution if and only if

$$F^h(1) > 0$$

Convergence

Let $\{\mathbf{e}_\kappa^*\}$ is sequence of solutions to approximated optimization problems $(\mathcal{P}_\kappa)$ and $u_h(\mathbf{e}_\kappa^*) \in \mathbb{V}_h$ is a solution to state problem $(P_h(\mathbf{e}_\kappa^*))$. Let the condition $L^h(1) > 0$ holds. Then there exists a subsequence $\{\mathbf{e}_{\kappa_j}^*\} \subset U_{ad,\kappa}, \ \mathbf{e}_\kappa$ such that

$$\mathbf{e}_{\kappa_j} \to \mathbf{e} \ \text{ in } U_{ad},$$

$$u_h(\mathbf{e}_{\kappa_j}) \to u \ \text{ in } \mathbb{V},$$

where $\mathbf{e}$, $u(\mathbf{e})$ is an optimal pair to the problem $(\mathcal{P})$.

Design variables

We can associate the design variables t_κ and q_κ with $(n(\kappa)+1)$ and $(n(h))$-dimensional vectors, respectively. Components of these vectors can be obtained as function values at nodal points Δ_κ.

Thickness

$$\mathcal{U}_\kappa^t = \left\{ \mathbf{t}_\kappa \in \mathbb{R}^{n(\kappa)+1} : \; T_0 \leq t_i \leq T_1, \; \sum_{i=1}^{n(\kappa)} \frac{\kappa}{2}(t_{i-1}+t_i) = T_2, \; \frac{|t_{i-1}-t_i|}{|b_i - b_{i-1}|} \leq T_3 \right\},$$

where the relations hold for $i = 0, \ldots, n(\kappa)$ and $i = 1, \ldots, n(\kappa)$, respectively.

Subsoil stiffness

$$\mathcal{U}_\kappa^q = \left\{ \mathbf{q}_\kappa \in \mathbb{R}^{n(\kappa)} : \quad Q_0 \leq q_i \leq Q_1, \; \forall i = 1, \ldots, n(\kappa) \right\}.$$

State problem

Using the finite element method, the approximated state problem $(P_h(\mathbf{e}_\kappa))$ takes a form of system of nonlinear algebraic equations.

System of nonlinear algebraic equations

$$\mathbb{K}\mathbf{u} + \mathbb{M}(\mathbf{u})\mathbf{u} = \mathbb{F}, \quad \mathbf{u}(\mathbf{e}_\kappa) \in \mathbb{R}^{2n(h)+2}$$

To solve this kind of problem we can use an approach based on the decomposition $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^- = \mathbf{v} - \mathbf{w}$.

Mixed linear complementarity problem

$$\begin{pmatrix} \mathbb{K} & \mathbf{S}^T & \mathbf{0}^T \\ \mathbf{S} & -\mathbf{D} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbb{F} \\ \mathbf{o} \end{pmatrix}$$

$$\mathbf{v}^T\mathbf{w} = 0, \quad \mathbf{v}, \mathbf{w} \geq 0.$$

This problem can be solved using Lemke method, interior point approach or Gauss-Seidel method with projection.

Optimization problem

The optimization problem may be expressed in the following algebraic form:

Nonlinear minimization problem

$$\mathbb{J}(\mathbf{e}, \mathbf{u}(\mathbf{e})) \to \min$$
$$\mathbf{A}\mathbf{e} \leq \mathbf{a}$$
$$\mathbf{B}\mathbf{e} = \mathbf{b}$$
$$\mathbf{c} \leq \mathbf{e} \leq \mathbf{d},$$

where $\mathbf{u}(\mathbf{e})$ is a solution to

$$\begin{pmatrix} \mathbb{K} & \mathbf{S}^T & \mathbf{0}^T \\ \mathbf{S} & -\mathbf{D} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbb{F} \\ \mathbf{o} \end{pmatrix}.$$

$$\mathbf{v}^T\mathbf{w} = 0, \quad \mathbf{v}, \mathbf{w} \geq 0.$$
$$\mathbb{J} : \mathbb{R}^{2n(\kappa)+1} \to \mathbb{R}.$$
$$\mathbf{A} \in \mathbb{R}^{n(\kappa)\times(2n(\kappa)+1)}, \; \mathbf{a} \in \mathbb{R}^{n(\kappa)}, \; \mathbf{c}, \mathbf{d} \in \mathbb{R}^{2n(\kappa)+1}.$$

Sensitivity analysis

Outline of the optimization problem

$$\mathbf{e} \longmapsto \mathbf{u}(\mathbf{e}) \longmapsto \mathbb{J}(\mathbf{e}, \mathbf{u}(\mathbf{e})).$$

Continuity is important, but it is not enough. To better understand the problem, other properties are needed. One of the most important properties is differentiability. It has been proved that $\mathbf{e} \longmapsto \mathbf{u}(\mathbf{e})$ is Lipschitz continuous as well as mappings $\mathbf{e} \longmapsto \mathbf{v}(\mathbf{e})$ and $\mathbf{e} \longmapsto \mathbf{w}(\mathbf{e})$. In fact these mappings are directionally differentiable at any $\mathbf{e} \in \mathcal{U}_\kappa$ and in any direction $\mathbf{d} \in \mathbb{R}^{2n(\kappa)+1}$.

Directional derivative

$$\begin{pmatrix} \mathbb{K} & \mathbf{S}^T & \mathbf{0}^T \\ \mathbf{S} & -\mathbf{D} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{u}}(\mathbf{e};\mathbf{d}) \\ \dot{\mathbf{v}}(\mathbf{e};\mathbf{d}) \\ \dot{\mathbf{w}}(\mathbf{e};\mathbf{d}) \end{pmatrix} = \begin{pmatrix} \mathbb{F} - \mathbb{K}'(\mathbf{e};\mathbf{d})\mathbf{u} - \mathbf{S}'^T(\mathbf{e};\mathbf{d})\mathbf{v} \\ \mathbf{D}'(\mathbf{e};\mathbf{d})\mathbf{v} - \mathbf{D}'(\mathbf{e};\mathbf{d})\mathbf{w} - \mathbf{S}'(\mathbf{e};\mathbf{d})\mathbf{u} \end{pmatrix}$$

$$\dot{\mathbf{v}}_i = 0 \quad i \in I_-, \; \dot{\mathbf{w}}_i = 0 \quad i \in I_+, \; \dot{\mathbf{v}}_i \dot{\mathbf{w}}_i = 0, \; \dot{\mathbf{v}}_i, \dot{\mathbf{w}}_i \geq 0 \quad i \in I_0.$$

Sensitivity analysis

Subgradient of the cost functional

$$\xi_{\mathbf{u}}^T(\mathbf{e})\nabla_{\mathbf{u}}\mathbb{J}(\mathbf{u}(\mathbf{e})) \in \partial\mathbb{J}(\mathbf{u}(\mathbf{e})), \quad \xi_{\mathbf{u}}^T(\mathbf{e}) \in \partial\mathbf{u}(\mathbf{e}).$$

We want to apply an adjoint state technique to eliminate $\xi_{\mathbf{u}}^T(\mathbf{e})$ in the previous expression.

Adjoint state problem

$$\begin{pmatrix} \mathbb{K} & \mathbf{S}^T & \mathbf{0}^T \\ \mathbf{S} & -\mathbf{D} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{p}(\mathbf{e}) \\ \mathbf{r}(\mathbf{e}) \\ \mathbf{s}(\mathbf{e}) \end{pmatrix} = \begin{pmatrix} \nabla_{\mathbf{u}}\mathbb{J}(\mathbf{u}(\mathbf{e})) \\ \mathbf{0} \end{pmatrix}$$

$$\mathbf{r}_i = 0 \quad i \in I_-, \ \mathbf{s}_i = 0 \quad i \in I_+, \ \mathbf{r}_i\mathbf{s}_i = 0, \ \mathbf{r}_i, \mathbf{s}_i \geq 0 \quad i \in I_0.$$

$$[\nabla\mathbb{F} - \nabla\mathbb{K}\mathbf{u} - \nabla\mathbf{S}^T\mathbf{v} - \mathbf{S}^T\xi_{\mathbf{v}}]^T\mathbf{p} + \xi_{\mathbf{u}}^T\mathbf{S}^T\mathbf{r} \in \partial\mathbb{J}(\mathbf{u}(\mathbf{e})).$$

$$\xi_{\mathbf{u}}^T\mathbf{S}^T\mathbf{r} - [\mathbf{S}^T\xi_{\mathbf{v}}]^T\mathbf{p} = 0 \Rightarrow [\nabla\mathbb{F} - \nabla\mathbb{K}\mathbf{u} - \nabla\mathbf{S}^T\mathbf{v}]^T\mathbf{p} \in \partial\mathbb{J}(\mathbf{u}(\mathbf{e})).$$

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Thank you for your attention.