Influence of Preference Information on Stability of Multiobjective Combinatorial Problems

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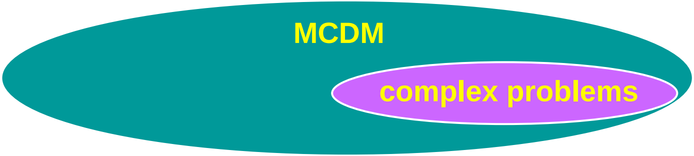
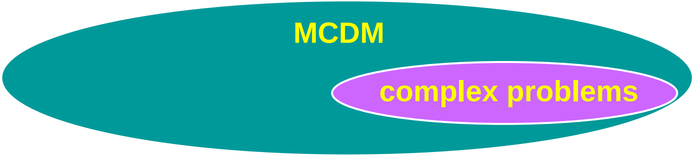


Reduce to a single objective optimization problem

Derive the Pareto set and let DM to choose the solution

Build an interactive method of MCDM



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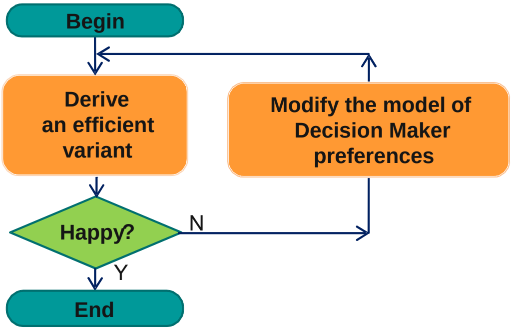
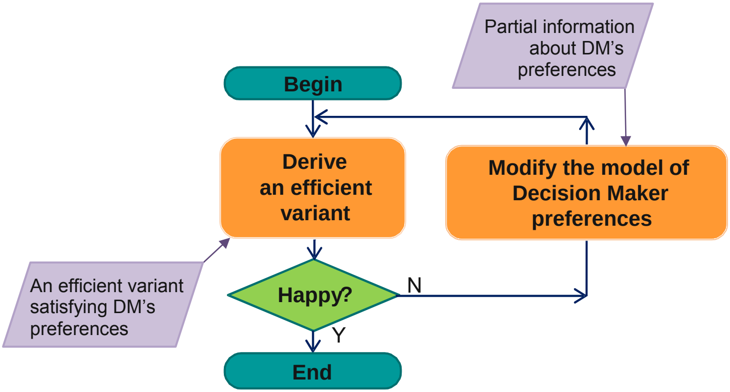
Interactive MCDM problem–solving method



Interactive MCDM problem–solving method



Multiple objective optimization problem

$$\max_{x \in X} f(x)$$

X is the set of feasible solutions (decision variants);

$f(x) = (f_1(x), f_2(x), \ldots, f_k(x))$ is the vector criterion function;

$f_i \colon X \to \mathbf{R}$, $i \in N_k$, are objective functions;

$N_k = \{1,2,\ldots,k\}$; $k \geq 2$.

variant $x \in X \rightarrow$ outcome (vector evaluation) $y = f(x) \in Y$.

$$\max_{y \in Y} y$$

No information about DM's preferences
(Pareto domination relation)

DM prefers to increase values of some objective functions, if no other objective function values are decreased.

Complete information about DM's preferences
(scalarizing function $f(y_1, y_2, \ldots, y_k)$ as an example)

One unit decrease of j-th objective function at y is substituted by $f'_j(y) / f'_i(y)$ units increase of i-th objective function.

Partial information about DM's preferences

DM agrees to trade a unit of j-th objective function

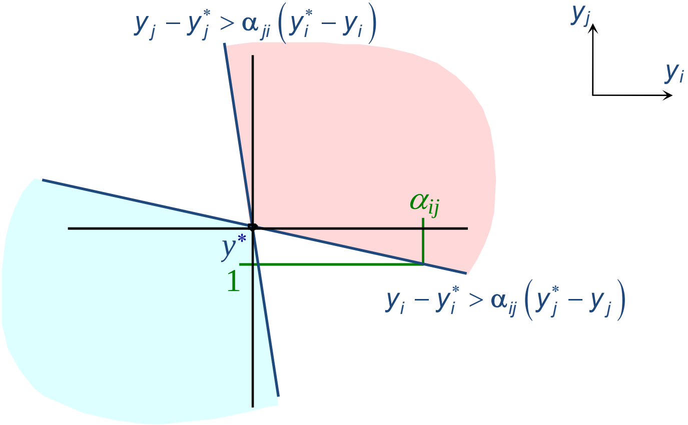
for at least α_{ij} units of i-th objective function.

DM agrees to trade at most β_{ij} units of j-th objective function

for one unit of i-th objective function.

$$1/\alpha_{ij} = \beta_{ij}$$

Partial information about DM's preferences



Efficient solutions satisfying partial DM's preferences

Let y^* be efficient. Trade-off coefficient t_{ij} is the maximal ratio $\left(y_i - y_i^*\right) / \left(y_i^* - y_i\right)$ which can be achieved for

$y \neq y^*$, $y \in S \subseteq Y$, where $y_i \geq y^*_i$, $y_j < y^*_j$.

Applying partial information about DM's preferences

For given efficient solution y^*:

for each pair of criteria i, j:

calculate trade-off coefficient t_{ij};

if $t_{ij} > \alpha_{ij}$ then y^* does not satisfy DM's preferences♣.

♣ because there exists y such that $y_i - y_i^* > \alpha_{ij}(y_j^* - y_j)$.

Efficient solutions satisfying partial DM's preferences (bounds on trade-off coefficients)

Denote $Z_j^{<}(y^*,Y) = \{\, y \in Y:\ y_j < y_j^* \ \&\ \forall\ s \in N_k \backslash \{j\}\ (y_s \geq y_s^*)\, \}$.

Let $i, j \in N_k$, $i \neq j$. If $Z_j^{<}(y^*,Y) \neq \varnothing$, then the number

$$T_{ij}(y^*,Y) = \sup_{y \in Z_j^{<}(y^*,Y)} \frac{y_i - y_i^*}{y_j^* - y_j}$$

is called **global trade-off coefficient** of efficient outcome y^* between i-th and j-th objective functions.

If $Z_j^{<}(y^*,Y) = \varnothing$, then by definition $T_{ij}(y^*,Y) = -\infty$.

Deriving efficient solutions satisfying bounds on trade-off coefficients

Let $y^0 \in \mathbf{R}^k$, $y_i^0 > y_i$ for all $y \in Y$, $i \in N_k$ and let $\rho_i > 0$, $i \in N_k$.
If for some $\lambda \in \mathbf{R}_>^k$, outcome y^* is a solution of

$$\min_{y \in Y} \left(\max_{i \in N_k} \lambda_i \left(y_i^0 - y_i \right) + \sum_{j \in N_k} \rho_j \left(y_j^0 - y_j \right) \right),$$

then y^* is efficient and

$$T_{ij}(y^*, Y) \le \frac{1 + \rho_j}{\rho_i} \quad \text{for all } \; i, j \in N_k, i \ne j \, .$$

I. Kaliszewski, W. Michalowski, Efficient solutions and bounds on tradeoffs. *Journal of Optimization Theory and Applications, 94 (1997), p. 381–394.*

Deriving efficient solutions satisfying bounds on trade-off coefficients

Let $Z - \mathbf{R}^k_{\geq}$ be convex.

If for some $\lambda \in \mathbf{R}^k_{>}$, outcome y^* is a solution of

$$\max_{y \in Y} \sum_{i \in N_k} \lambda_i y_i \ ,$$

then $T_{ij}(y^*, Y) \leq \lambda_j / \lambda_i$ for any $i, j \in N_k,\ i \neq j$.

I. Kaliszewski, S. Zionts, A generalization of the Zionts-Wallenius multiple criteria decision making algorithm. *Control and Cybernetics, 33 (2004), No. 3, p. 477-500.*

Drawbacks of the global trade-off approach

- only outcomes from subset $Z_j^{<}(y^*, Y)$ are considered;
- each relation $(y_i - y_i^*)/(y_j^* - y_j) \leq \alpha_{ij}$ is verified separately;
- $k \cdot (k-1)$ bounds (α_{ij} or β_{ij}) $\leftrightarrow$ k scalarizing parameters.

New approach to modeling partial DM’s preferences: asymmetricity issue

Let DM agree to trade 1 unit of i for α_{ij} units of j.

Then DM will not agree to

trade α_{ij} units of j for less than 1 unit of i

or trade 1 unit of j for less than $1/\alpha_{ij}$ units of i.

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It follows that

$$\alpha_{ij}\alpha_{ji} \geq 1 \text{ for any } i \neq j;$$

$$\beta_{ij}\beta_{ji} \leq 1 \text{ for any } i \neq j.$$

New approach to modeling partial DM's preferences: transitivity issue

Let DM agree to trade 1 unit of i for α_{is} units of s
and to trade α_{is} units of s for $\alpha_{is}\alpha_{sj}$ units of j.

Then DM will agree to trade 1 units of i for $\alpha_{is}\alpha_{sj}$ units of j.

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It follows that

$$\alpha_{is}\alpha_{sj} \geq \alpha_{ij} \quad \text{for any } i \neq s,\ j \neq s;$$

$$\beta_{is}\beta_{sj} \leq \beta_{ij} \quad \text{for any } i \neq s,\ j \neq s.$$

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$$\alpha_{is}\alpha_{sj} \geq \alpha_{ij} \quad \text{for any } i \neq s,\ j \neq s;$$

$$\beta_{is}\beta_{sj} \leq \beta_{ij} \quad \text{for any } i \neq s,\ j \neq s.$$

$$\Downarrow \ (\text{when } i = j,\ \beta_{ii} := 1)$$

$$\beta_{is}\beta_{si} \leq 1 \text{ for any } i \neq j.$$

New approach to modeling partial DM's preferences

Let $\mathrm{B}=[\beta_{ij}]_{kxk}$, where $\beta_{ii} = 1$ for any $i \in N_k$.

Variant x^* (outcome y^*) is called B-efficient,
if it is weakly efficient in the problem

$$\max_{x \in X} \mathrm{B} f(x)$$

$$\left(\max_{y \in Y} \mathrm{B} y \right).$$

New approach to modeling partial DM's preferences

Theorem. Let $\beta_{is}\beta_{sj} \leq \beta_{ij}$ for any $i \neq s$, $j \neq s$.

Then any weakly efficient solution of

$$\max_{x \in X} \mathrm{B} f(x)$$

satisfies following bounds on global trade-off coefficients:

$$T_{ij}(y^*, Y) \leq 1/\beta_{ij}.$$

Comparing B-efficiency approach and bounding global trade-off coefficients

y^* does not satisfy DM's preferences, if for some $y \in Z_j^<(y^*, Y)$...

$y_j^* - y_j < \beta_{ij}(y_i - y_i^*)$ for some i

Check if the price of decreasing in j-th objective function is small enough to pay for increasing some other objective function.

global trade-off approach

$$y_j^* - y_j < \sum_{i \in N_k \setminus \{j\}} \beta_{ij} \left(y_i - y_i^* \right)$$

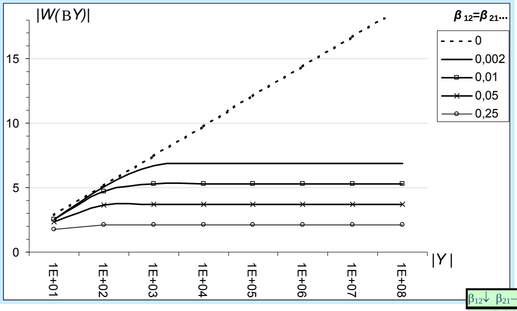
“Include in the bill” all the gains in other objective functions.

new approach

New approach to modeling partial DM's preferences: further research

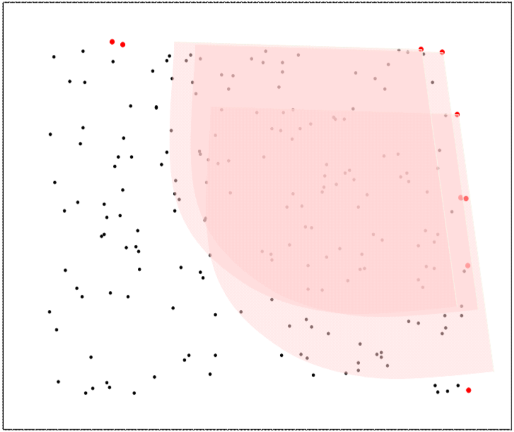
- Interpretations of B-efficiency (in comparison with bounds on global trade-off coefficients and other approaches).
- B-efficiency and proper efficiency; algebraic properties.
- Applying B-efficiency in decision making algorithms (e.g. instead of "Chebyshev's scalarizing function with a regularization term").
- Trade-off approach in modeling altruistic game equilibria.
- Influence of β-coefficients on the average number of efficient solutions in multiobjective discrete optimization problems.

Let Y be finite, y_1 and y_2 be independent uniformly distributed random numbers. Then $|P(Y)|$ grows as $\ln |Y|$.



$\beta_{12}\downarrow$ $\beta_{21}\rightarrow$	***0,002***	***0,01***	***0,05***	***0,25***
0,002	6,91	6,10	5,30	4,49
0,01	6,10	5,30	4,49	3,69
0,05	5,30	4,49	3,69	2,90
0,25	4,49	3,69	2,90	2,13

$k = 2, \quad |Y| = 200$



Stability of multiobjective 0-1 programming problems

$$Cx \to \max_{x \in X},$$

$X \subseteq \{0,1\}^n, n > 1,$ is the set of feasible solutions, $|X| > 1$;

$C = [c_{ij}]_{k \times n} \in \mathbf{R}^{k \times n}$ is the matrix of vector criterion coefficients.

$P(C) = \{x \in X : \pi(x,C) = \varnothing\}$ is the set of Pareto optimal solutions,

$\pi(x,C) = \{x' \in X : Cx' \geq Cx, Cx' \neq Cx\}$.

$\|z\|_\infty = \max_{i \in N_q} |z_i|$, $\|z\|_1 = \sum_{i \in N_q} |z_i|$ are norms for any $z \in \mathbf{R}^q$, $q \in \mathbf{N}$.

Stability radius of the Pareto problem

$\Delta(\varepsilon) = \{D \in \mathbf{R}^{k\times n} : \|D\|_\infty < \varepsilon\}$ is the set of perturbing matrices.

The problem is called stable, if there exists $\varepsilon > 0$ such that

$$\forall D \in \Delta(\varepsilon)\left(P(C+D) \subseteq P(C)\right).$$

The stability radius of the problem is defined by

$$\rho(C) = \begin{cases} \sup\, \Omega(C), & \text{if } \Omega(C) \neq \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

where $\Omega(C) = \left\{ \varepsilon > 0 : \forall D \in \Delta(\varepsilon)\ \left(P(C+D) \subseteq P(C)\right) \right\}$.

Applications of the stability radius

Let $Cx \to \max_{x \in X}$ be a model of a practical problem.

Suppose that in fact the problem parameters are uncertain and the real problem is described by $(C + D)x \to \max_{x \in X}$.

Applications of the stability radius

Let $Cx \to \max_{x \in X}$ be a model of a practical problem.

Suppose that in fact the problem parameters are uncertain and the real problem is described by $(C + D)x \to \max_{x \in X}$.

If $||D||_\infty < \rho(C)$, then the set of efficient solutions of the real problem includes in the set of efficient solutions of the solved problem. Thus we are guaranteed that no efficient solution of the real problem is missed.

Applications of the stability radius

Let one need to solve a series of instances of a computationally hard problem.

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Formulate the solved problem instance as the initial problem, the unsolved instance as the perturbed problem, where the perturbations are differences between parameters of the problem instances.

Applications of the stability radius

Let one need to solve a series of instances of a computationally hard problem.

Consider two problem instances; one of which has already been solved and the other instance is unsolved yet.

Formulate the solved problem instance as the initial problem, the unsolved instance as the perturbed problem, where the perturbations are differences between parameters of the problem instances.

If these differences are less than the stability radius, then there is a chance that the latter instance may have the same solution as the former instance.

Formula of the stability radius

Let ♣♥ denote the ♥-th row of matrix ♣.

If $P(C) = X$ then $\rho(C) = \infty$. Otherwise

$$\rho(C) = \min_{x \in \bar{P}(C)} \max_{x' \in \pi(x,C)} \min_{i \in N_k} \frac{C_i(x' - x)}{\| x' - x \|_1},$$

where $\bar{P}(C) = X \setminus P(C)$.

V.A. Emelichev, E. Girlich, Yu.V. Nikulin, D.P. Podkopaev. Stability and regularization of vector problems of integer linear programming. *Optimization, 51 (2002), p. 645–676.*

Formula of the stability radius

If $P(C)=X$ then $\rho(C)=\infty$. Otherwise

$$\rho(C)=\min_{x\in\bar{P}(C)}\ \max_{x'\in\pi(x,C)}\ \min_{i\in N_k}\frac{C_i(x'-x)}{\|x'-x\|_1}.$$

for any $x\in\bar{P}(C)$

there exists $x'\in\pi(x,C)$

such that for any $i\in N_k$

we have $(C_i+D_i)(x'-x)\geq 0$

Calculating stability radius (of an efficient solution in the case $k = 1$)

Theorem. Let x be an optimal solution to $cx \to \max_{x \in X}$. The stability radius of x is the maximal ψ satisfying

$$\min_{x' \in X \setminus \{x\}} \left\{ \sum_{i \in N_n} \left(-c_i - \psi d_i\right) x'_i \right\} \geq \sum_{i \in N_n} \left(-c_i + \psi\right) x_i$$

where $d_i = \begin{cases} 1 & \text{if } x_i = 0, \\ -1 & \text{if } x_i = 1. \end{cases}$

N. Chakravarti, A. Wagelmans, Calculation of stability radius for combinatorial optimization problems. *Operations Research Letters, 23 (1999), p. 1–7*.

Calculating stability radius
(of an efficient solution in the case $k = 1$)

$$\rho(x,c) = \min_{x' \in X \setminus \{x\}} \frac{\langle c, (x - x') \rangle}{\| x - x' \|_1}$$

$$\Downarrow$$

$$\min_{x' \in X \setminus \{x\}} \left\{ \sum_{i \in N_n} (-c_i - \psi d_i) x'_i \right\} \geq \sum_{i \in N_n} (-c_i + \psi) x_i$$

V. Emelichev, D. Podkopaev, Quantitative stability analysis for vector problems of 0-1 programming. *Discrete Optimization, 7 (2010), p. 48-63.*

Stability radius of B-efficient solutions

$P_{\mathrm{B}}(C) = \{x \in X : \pi_{\mathrm{B}}(x,C) = \varnothing\}$ is the set of B-efficient solutions,
$\pi_{\mathrm{B}}(x,C) = \{x' \in X : \mathrm{B}Cx' > \mathrm{B}Cx\}$.

The problem is called stable, if there exists $\varepsilon > 0$ such that

$$\forall D \in \Delta(\varepsilon) \left(P_{\mathrm{B}}(C+D) \subseteq P_{\mathrm{B}}(C)\right).$$

The stability radius of the problem is defined by

$$\rho_{\mathrm{B}}(C) = \begin{cases} \sup\, \Omega_{\mathrm{B}}(C), & \text{if } \Omega_{\mathrm{B}}(C) \neq \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

where $\Omega_{\mathrm{B}}(C) = \left\{ \varepsilon > 0 : \forall D \in \Delta(\varepsilon) \ \left(P_{\mathrm{B}}(C+D) \subseteq P_{\mathrm{B}}(C)\right) \right\}$.

Formula of the stability radius of B-problem

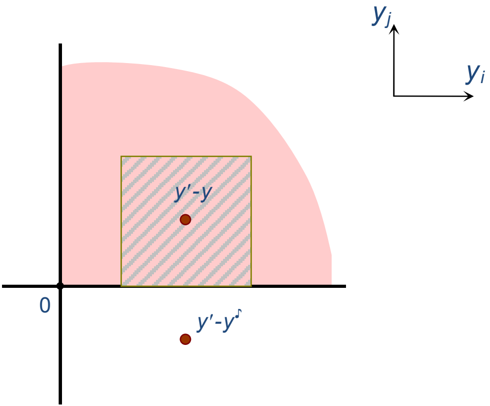
If $P_{\mathrm{B}}(C)=X$ then $\rho_{\mathrm{B}}(C)=\infty$. Otherwise

$$\rho_{\mathrm{B}}(C)=\min_{x\in\bar{P}_{\mathrm{B}}(C)}\ \max_{x'\in\pi_{\mathrm{B}}(x,C)}\ \min_{i\in N_k}\frac{\mathrm{B}_iC(x'-x)}{\|\mathrm{B}\|_1\|x'-x\|_1},$$

where $\bar{P}_{\mathrm{B}}(C)=X\setminus P_{\mathrm{B}}(C)$.

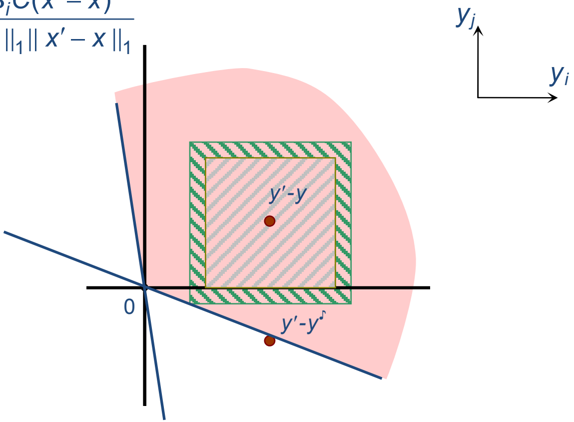
Influence of B on the stability radius

$$\min_{i\in N_k} \frac{C_i(x'-x)}{\| x'-x \|_1}$$



Influence of B on the stability radius

$$\min_{i \in N_k} \frac{\mathrm{B}_i C(x' - x)}{\| \mathrm{B} \|_1 \| x' - x \|_1}$$



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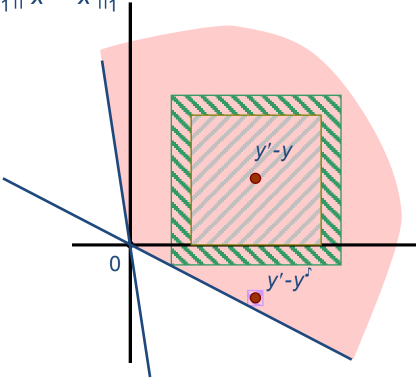
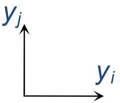
$$\min_{i \in N_k} \frac{\mathrm{B}_i C(x' - x)}{\| \mathrm{B} \|_1 \| x' - x \|_1}$$



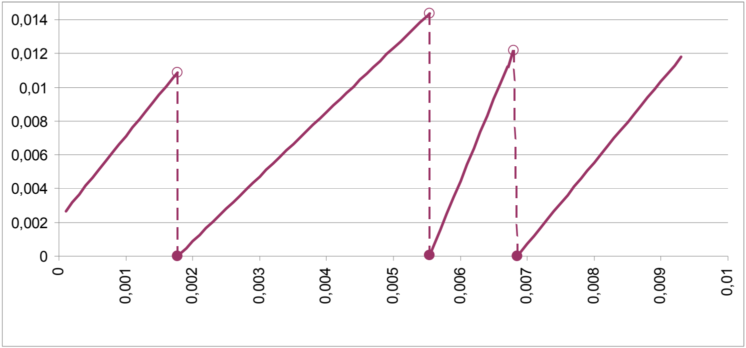


Influence of B on the stability radius: an example

TSP: $k = 5$, $n = 6 \Rightarrow |X| = 720$; c_{ij} = NormalDistribution[5,5], Mathematica 6.0.

Influence of B on the stability radius: an example

TSP: $k = 5$, $n = 6 \Rightarrow |X| = 720$; c_{ij} = NormalDistribution[5,5]



Thank you for your attention

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