On stability of a lexicographic optimum of the multicriteria combinatorial center and median location problems

Olga Karelkina, University Of Turku, 2010

Outline

- Problem statement
- Location problems
- Lexicographic dominance
- Lexicographic optimum stability
- Definitions of sets and binary relations
- Stability criteria (center location)
- Stability criteria (median location)
- References

Let us consider these location problems in the following formulation

$N_m = \{1, 2, \ldots, m\}$ the set of possible facility (supplier) locations

N_n the set of consumer (client) locations

$C = (c_{ijk}) \in \text{¡}^{m \times n \times s}$ costs (distance etc.) matrix

T is the set of nonempty subsets of $N_m, T \subset 2^{N_m}, |T| \geq 2$

Vector function

$$f(t, C) = (f_1(t, C), f_2(t, C), \ldots, f_s(t, C))$$

Multicriteria center location problem $Z_1^s(C)$

$$f_k(t,C) = \max_{j \in N_n} \min_{i \in t} c_{ijk} \rightarrow \min_{t \in T}, \ \ k \in N_s$$

Multicriteria median location problem $Z_2^s(C)$

$$f_k(t,C) = \sum_{j \in N_n} \min_{i \in t} c_{ijk} \rightarrow \min_{t \in T}, \ \ k \in N_s$$

The p-center problem (scalar case s=1)

$$\max_{j \in N_n} \min_{i \in t} c_{ij} \to \min$$

The p-median problem (scalar case s=1)

$$\sum_{j \in N_n} \min_{i \in t} c_{ij} \to \min$$

$$t \in T, |t| = p, \ 1 \leq p \leq m-1$$

The binary relation of lexicographic domination

$$t \succ_C t' \Leftrightarrow \exists r \in N_s \ (f_r(t,C) > f_r(t',C) \ \& \ r = \min\{k \in N_s : f_k(t,C) \neq f_k(t',C)\})$$

The lexicographic set

$$L^s(C) = \{t \in T : \forall t' \in T \quad (t \succeq_C t')\}$$

$$t \succeq_C t' \Leftrightarrow (f(t,C) = f(t',C)) \vee$$

$$\vee \exists r \in N_s \quad (f_r(t,C) < f_r(t',C) \ \& \ r = \min\{k \in N_s \ (f_k(t,C) \neq f_k(t',C))\})$$

$$L_k^s(C) = \text{Argmin}\{f_k(t,C) \mid t \in L_{k-1}^s(C)\}, \quad k \in N_s$$

$$T = L_0^s(C) \supseteq L_1^s(C) \supseteq L_2^s(C) \supseteq \ldots \supseteq L_s^s(C) = L^s(C)$$

$$\Omega(\varepsilon) = \{C' \in \text{¡}^{m \times n \times s} : \|C'\| < \varepsilon\},$$

$$\|C'\| = \max\{\, |c'_{ijk}| : (i, j, k) \in N_m \times N_n \times N_s \}$$

Lexicographic optimum $t \in L^s(C)$ of the problem $Z_1^s(C)$

$\left(Z_2^s(C)\right)$ is called stable if

$$\exists \varepsilon > 0 \;\; \forall C' \in \Omega(\varepsilon) \;\; (t \in L^s(C + C'))$$

$$g_{jk}(t,C) = \min\{c_{ijk} : i \in t\}$$

$$N_{jk}(t,C) = \{l \in t : f_k(t,C) = g_{jk}(t,C) = c_{ljk}\}$$

$$J_k(t,C) = \{j \in N_n : f_k(t,C) = g_{jk}(t,C)\}$$

$$t \underset{C,k}{\vdash\!\!-} t' \Leftrightarrow t \underset{C,k}{|\!:} t' \underset{C,k}{|\!\approx} t$$

$$t \underset{C,k}{|\!:} t' \Leftrightarrow \forall j \in J_k(t,C) \ \ (N_{jk}(t,C) \supseteq N_{jk}(t',C))$$

$$t' \underset{C,k}{|\!\approx} t \Leftrightarrow J_k(t',C) \supseteq J_k(t,C)$$

Theorem 1 A lexicographic optimum $t^0 \in L^s(C)$ of the center location problem $Z_1^s(C)$ is stable if and only if

$$\forall k \in N_s \quad \forall t \in L_k^s(C) \quad (t^0 \underset{C,k}{\vdash\!\!-} t)$$

$$t^0 \underset{C,k}{\vdash\!\!-} t \Leftrightarrow \forall j \in J_k(t^0, C) \subseteq J_k(t, C) \quad \left(N_{jk}(t^0, C) \supseteq N_{jk}(t, C)\right)$$

Proof. Necessity. Lemma 1, Lemma 2 .

Lemma1 *If* $t^0 \in L^s(C)$ *and there exist* $r \in N_s$ *and* $t \in L_r^s(C)$ *such that* $t \underset{C,r}{\overline{\approx}} t^0$, *then the solution* t^0 *is not stable.*

$$\exists\ q \in J_r(t^0, C) \setminus J_r(t, C), \exists\ p \in t \setminus t^0, \varepsilon > 0, 0 < \alpha < \varepsilon$$

$$c'_{ijk} = \begin{cases} \alpha, & \text{if } i \in t^0, j = q, k = r, \\ 0 & \text{otherwise} \end{cases}$$

Lemma2 *If* $t^0 \in L^s(C)$ *and there exist* $r \in N_s$ *and* $t \in L_r^s(C)$ *such that* $t^0 \overline{\underset{C,r}{|:}}\ t$, *then the solution* t^0 *is not stable.*

$t \underset{C,r}{|\approx} t^0, \exists\ q \in J_r(t^0, C) \subseteq J_r(t, C), \exists\ p \in N_{qr}(t, C) \setminus N_{qr}(t^0, C),$

$\varepsilon > 0, 0 < \alpha < \varepsilon$

$$c'_{ijk} = \begin{cases} -\alpha, & \text{if } i = p, j = q, k = r, \\ -\alpha, & \text{if } i \in t, j \in N_n \setminus \{q\}, k = r, \\ 0 & \text{otherwise} \end{cases}$$

Sufficiency. Case 1: $t \in L_1^s(C)$

- $t \in L^s(C)$, Pr1
- $t \in L_1^s(C) \setminus L^s(C)$, Pr2(ii)

Case 2: $t \in T \setminus L_1^s(C)$, Pr2(i)

Pr1 *If* $\forall k \in N_s \;\; t \underset{C,k}{\vdash\!\!-\!-} t'$, *then*

$$\exists \varepsilon > 0 \;\; \forall C' \in \Omega(\varepsilon) \;\; \forall k \in N_s \;\; \left(t \underset{C+C'}{\text{Ŧ}} t' \right)$$

Pr2 *If any of the following conclusions holds*

$(i)\; f_1(t', C) > f_1(t, C)$

$(ii)\; \exists r \in N_{s-1} \;\; (f_{r+1}(t', C) > f_{r+1}(t, C) \;\&\; \forall k \in N_r \;\; (t \underset{C,k}{\vdash\!\!-\!-} t'))$

then $\exists \varepsilon > 0 \;\; \forall C' \in \Omega(\varepsilon) \;\; (t \underset{C+C'}{\text{Ŧ}} t')$

Theorem 2. A lexicographic optimum $t^0 \in L^s(C)$ of the median location problem $Z_2^s(C)$ is stable if and only if

$$\forall k \in N_s \quad \forall t \in L_k^s(C) \quad \forall j \in N_n \quad \left(N_{jk}(t^0,C) \supseteq N_{jk}(t,C)\right)$$

$$N_{jk}(t,C) = \mathrm{Argmin}\{c_{ijk} : i \in t\}$$

Proof. Necessity.

$$\exists q \in N_n \quad (N_{qr}(t^0,C) \not\supseteq N_{qr}(t^*,C)), \quad \exists l \in N_{qr}(t^*,C) \setminus N_{qr}(t^0,C)$$

$$c'_{ijk} = \begin{cases} -\alpha, & \text{if } (i,j,k) = (l,q,r), \\ 0 & \text{otherwise} \end{cases}$$

Sufficiency. Use the same reasoning as in Theorem 1

Pr3 *If* $\forall k \in N_s \;\; \forall j \in N_n \;\; (N_{jk}(t,A) \supseteq N_{jk}(t',A))$,

then $\exists \varepsilon > 0 \; \forall C' \in \Omega(\varepsilon) \; \forall k \in N_s \; \left(t \underset{C+C'}{\bar{\mathrm{F}}} t' \right)$

Pr4 *If any of the following conditions holds*

$(i) \; f_1(t,C) > f_1(t^0,C)$

$(ii) \; \exists r \in N_{s-1} \;\; (f_{r+1}(t,C) > f_{r+1}(t^0,C) \; \&$

$\& \; \forall k \in N_r \;\; \forall j \in N_n \;\; (N_{jk}(t^0,A) \supseteq N_{jk}(t,A))$

then $\exists \varepsilon > 0 \;\; \forall C' \in \Omega(\varepsilon) \;\; (t \underset{C+C'}{\mathrm{f}} t^0)$

1-center (median) problem

C1 *If* $|t| = 1$ *for any* $t \in T$, *then the equality* $L_1^s(C) = \{t^0\}$ *is a necessary and sufficient condition of stability of* $t^0 \in L^s(C)$

C2 $t^0 \in L^s(C)$ *of the problem with partial criteria MINMIN* $(n = 1)$ *is stable if and only if*

$$\forall k \in N_s \quad \forall t \in L_k^s(C) \quad \left(N_k(t^0, C) \supseteq N_k(t, C)\right)$$

$$N_k(t, C) = \mathrm{Argmin}\{c_{ik} : i \in t\},\ C = (c_{ik}) \in \mathbb{R}^{m \times s}$$

REFERENCES

1. Christofides, N. (1975) Graph theory. An algorithmic approach. Academic Press, New York.
2. Ehrgott, M. (2005) Multicriteria optimization. Second edition. Springer, Berlin-Heidelberg.
3. Emelichev V.A., Kuzmin K.G. (2008) Stability criteria in vector combinatorial bottleneck problems in terms of binary relations. Cybernetics and systems analysis, V.44. N 3. P. 397--404.
4. Chakravarti, N. and Wagelmans, A. P. M. (1998) Calculation of stability radii for combinatorial optimization problem. Operations Research Letter, V.23. N 1, P. 1--7.
5. Emelichev, V. A. and Stepanishina, Yu. V. (2000) Quasistability of a vector nonlinear trajectory problem with the Pareto optimality principle. Russian Mathematics (Iz. VUZ), V.44. N 12, P. 25--30.
6. Emelichev, V. A., Girlich, E. and Karelkina, O. V. (2009) Postoptimal analysis of multicriteria combinatorial center location problem. Bulletin of Academy of Sciences of Moldova, Mathematics, N 3, P. 13--29.
7. Emelichev, V. A. and Karelkina, O. V. (2009) On quasistability of a lexicographic MINSUM arrangement problem. Discretnyi analis i issledovanie operacii.,V.16 . N 2, P. 74--84 (in Russian).

THANK YOU FOR YOUR INTEREST