Characteristic measures of fuzzy numbers

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Outline

Probability vs Possibility

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- Cheesman: 'The numerous schemes for representing and reasoning about uncertainty that have appeared in the AI literature are unnecessary, probability is all that is needed'.
- L. Zadeh: 'Probability theory should be based on fuzzy logic'.

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- Weaver's midcentury expectations on the progress in science and technology seem to be anticipating important topics in the field of Soft Computing (SC) and Computational Intelligence: vague, fuzzy or approximate reasoning, the meaning of concepts.
- In this article at the end of the 1940's Weaver mentioned may be for the first time at all a trichotomy of scientific problems.

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- A problem of disorganized complexity 'is a problem in which the number of variables is very large, and one in which each of the many variables has a behavior which is individually erratic, or perhaps totally unknown. However, in spite of this helter-skelter, or unknown, behavior of all the individual variables, the system as a whole possesses certain orderly and analyzable average properties'.

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- 'The whole question of evidence and the way in which knowledge can be inferred from evidence are now recognized to depend on these same statistical ideas, so that probability notions are essential to any theory of knowledge itself.'

A new class of problems

- In addition to, and in-between, the problems of simplicity and the problems of disorganized complexity he identified another kind of scientific problems: One is tempted to oversimplify, and say that scientific methodology went from one extreme to the other, from two variables to an astronomical number, and left untouched a great middle region.

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- The importance of this middle region, moreover, does not depend primarily on the fact that the number of variables involved is moderate.
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- The really important characteristic problems of this middle region, which science has as yet little explored or conquered, lies in the fact that these problems, as contrasted with the disorganized situations which statistics can cope, show the essential feature of organization.
- In fact, one can refer to this group of problems as those of organized complexity.

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- With regard to these problems Weaver stressed that the involved variables are all interrelated in a complicated, but nevertheless not in helter-skelter, fashion that these complex systems have parts in close interrelations, and that something more is needed than the mathematics of averages.
- These problems in the biological, medical, psychological, economic, and political sciences are just too complicated to yield to the old nineteenth-century techniques . . . and these new problems, moreover, cannot be handled with the statistical techniques so effective in describing average behaviour in problems of disorganized complexity.

Probability vs Possibility

Definition (Normalized monotone measure)

Let X be a non-empty set and $\mathcal{C}$ any σ-algebra of its subsets, then a set function $m : \mathcal{C} \rightarrow [0, 1]$ is a normalized monotone measure if it satisfies

1. $m(\emptyset) = 0, m(X) = 1$
2. $A \subseteq B \Rightarrow m(A) \leq m(B) \forall A, B \in \mathcal{C}$

Definition (Probability measure)

A probability measure, Pr, is an additive normalized monotone measure, i.e.

$$\mathrm{Pr}(\mathrm{A} \cup \mathrm{B}) = \mathrm{Pr}(\mathrm{A}) + \mathrm{Pr}(\mathrm{B})$$

for any disjoint subsets A and B of the event space.

Note that probability is sufficient to describe the likelihood of an event thanks to the autoduality property, i.e. $\mathrm{Pr}(\mathrm{A}) = 1 - \mathrm{Pr}(\bar{\mathrm{A}})$.

Probability vs Possibility

Definition (Possibility)

A (normalized) possibility measure, Pos, is an maxitive normalized monotone measure, i.e.

$$\mathrm{Pos}(\bigcup_{i \in I} A_i) = \sup_i \mathrm{Pos}(A_i)$$

for any family $A_i | A_i \in P(X), i \in I$, where I is an arbitrary index set.

But it is not autodual; thus we need another function

Definition (Necessity measure)

A necessity function, Nec, is defined as follows

$$\mathrm{Nec}(\bigcap_{i \in I} A_i) = \inf_i \mathrm{Nec}(A_i)$$

for any family $A_i | A_i \in P(X), i \in I$, where I is an arbitrary index set.

A problem

Example (Dubois and Prade)

... a professional gambler will distribute his stakes evenly if he knows that all the options on which he is betting have equal strength. In the absence of any information, the neophyte will do the same, because it is the most popular strategy. Subjective probabilities allow no distinction between these two states of knowledge and seems ill adapted to situations where the knowledge is sparse.

We have to make distinction between two types of uncertainty

1. Uncertainty due to variability of observations.
2. Uncertainty due to incomplete information.

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- If X and Y are random variables with probability density functions $f_X(x)$ and $f_Y(y)$, respectively, then the density function, $f_{X,Y}(x,y)$, of their joint random variable (X,Y), should satisfy the following properties

$$\int_{\mathbb{R}} f_{X,Y}(x,t)\mathrm{d}t = f_X(x), \int_{\mathbb{R}} f_{X,Y}(t,y)\mathrm{d}t = f_Y(y)$$

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- Furthermore, $f_X(x)$ and $f_Y(y)$ are called the the marginal probability density functions of random variable (X,Y).

Characteristics

- The expected (or mean) value of random variable X is defined as

$$M(X) = \int_{\mathbb{R}} x f_X(x) \mathrm{d}x,$$

and if g is a function of X then the expected value of $g(X)$ can be computed as

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- The variance of random variable X is defined by $\mathrm{cov}(X,X)$

$$\mathrm{var}(X) = \mathrm{cov}(X,X) = M((X - M(X))^2) = M(X^2) - M(X)^2.$$

Characteristics

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$$\mathrm{cor}(X, Y) = \frac{\mathrm{cov}(X, Y)}{\sqrt{\mathrm{var}(X)\mathrm{var}(Y)}}.$$

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- If we have auniform distribution U on the interval $[a,b]$, then the mean value is equal to

$$M(U)=\frac{a+b}{2},$$

and the variance is

$$M(U)=\frac{(b-a)^2}{12},$$

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- In classical set theory, a subset A of a set X can be defined by its characteristic function χ_A as a mapping from the elements of X to the elements of the set $\{0, 1\}$.
- The statement $'\mathrm{x\ is\ in\ A}'$ is true if the second element of the ordered pair is 1, and the statement is false if it is 0.

Origin

- Similarly, a fuzzy subset A of a set X can be defined as a set of ordered pairs, each with the first element from X, and the second element from the interval $[0, 1]$, with exactly one ordered pair present for each element of X.

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- This defines a mapping, μ_A, between elements of the set X and values in the interval $[0, 1]$. The value zero is used to represent complete non-membership, the value one is used to represent complete membership, and values in between are used to represent intermediate degrees of membership.
- It should be noted that the terms membership function and fuzzy subset get used interchangeably. $(A(x) := \mu_A(x))$

Basic definitions

- A γ-level set of a fuzzy set A in $\mathbb{R}^m$ is defined by $[A]^\gamma = \{x \in \mathbb{R}^m : A(x) \geq \gamma\}$ if $\gamma > 0$ and $[A]^\gamma = \text{cl}\{x \in \mathbb{R}^m : A(x) > \gamma\}$ (the closure of the support of A) if $\gamma = 0$.

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- We will use a subclass of fuzzy sets:

Definition

A fuzzy number is a convex fuzzy set on the real line with bounded support $\mathbb{R}$ such that

1. $\exists x_0 \in \mathbb{R}, \mu_A(x_0) = 1$
2. μ_A is piecewise continuous

(The convexity means that all the γ-level sets are convex.) Furthermore, we call $\mathcal{F}$ the family of all fuzzy numbers.

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- $[A]^\gamma$ is a closed convex subset of $\mathbb{R}$ for all $\gamma \in [0, 1]$. We use the notations

$$a_1(\gamma) = \min[A]^\gamma, \ a_2(\gamma) = \max[A]^\gamma$$

for the left-hand side and right-hand side of the $\gamma-$cut, respectively.

Basic definitions

- Fuzzy numbers can be considered as possibility distributions: if $A \in \mathcal{F}$ is a fuzzy number and $x \in \mathbb{R}$ a real number then $A(x)$ can be interpreted as the degree of possiblity of the statement $'\mathrm{x\ is\ A}'$. The possibility that A takes its value from $[a, b]$ is defined by

$$\mathrm{Pos}(A \in [a, b]) = \max_{x \in [a,b]} A(x).$$

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- A fuzzy set C in $\mathbb{R}^2$ is said to be a joint possibility distribution of fuzzy numbers $A, B \in \mathcal{F}$, if it satisfies the relationships

$$\max\{y \in \mathbb{R} \mid C(x, y)\} = A(x) \text{ and } \max\{x \in \mathbb{R} \mid C(x, y)\} = B(y),$$

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  for all $x, y \in \mathbb{R}$.
- A and B are called the marginal possibility distributions of C. Marginal possibility distributions are always uniquely defined from their joint possibility distribution.

Joint distribution

- Let C be a joint possibility distribution with marginal possibility distributions $A, B \in$, and let $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ and $[B]^\gamma = [b_1(\gamma), b_2(\gamma)]$, $\gamma \in [0, 1]$.

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- Then A and B are said to be non-interactive if their joint possibility distribution is $A \times B$,

$$C(x, y) = \min\{A(x), B(y)\},$$

for all $x, y \in \mathbb{R}$, which can be written in the form, $[C]^\gamma = [A]^\gamma \times [B]^\gamma$, that is, $[C]^\gamma$ is rectangular subset of $\mathbb{R}^2$, for any $\gamma \in [0, 1]$.

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  for all $x, y \in \mathbb{R}$, which can be written in the form, $[C]^\gamma = [A]^\gamma \times [B]^\gamma$, that is, $[C]^\gamma$ is rectangular subset of $\mathbb{R}^2$, for any $\gamma \in [0,1]$.
- If A and B are are non-interactive then for any $x \in [A]^\gamma$ and any $y \in [B]^\gamma$ we have that the ordered pair (x, y) will be in $[C]^\gamma$ for any $\gamma \in [0,1]$.

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- Another extreme situation is when $[C]^\gamma$ is a line segment in $\mathbb{R}^2$.
- If one takes a point, x, from the γ-level set of A then one can take only $y = x$ from the γ-level set of B for the pair (x, y) to belong to $[C]^\gamma$.
- This point-to-point interactivity relation is the strongest one that we can envisage between γ-level sets of marginal possibility distributions.

Idea

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- In possibility theory we can use the principle of *average value* of appropriately chosen real-valued functions to define mean value, variance, covariance and correlation of possibility distributions.
- A function $f\colon [0,1] \to \mathbb{R}$ is said to be a weighting function if f is non-negative, monotone increasing and satisfies the following normalization condition $\int_0^1 f(\gamma)d\gamma = 1$.
- Different weighting functions can give different (case-dependent) importances to level-sets of possibility distributions.

Possibilistic mean

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- The f-weighted *possibilistic mean value* of $A \in \mathcal{F}$, with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0,1]$, is defined by

$$E_f(A) = \int_0^1 M(U_\gamma) f(\gamma) \mathrm{d}\gamma = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} f(\gamma) \mathrm{d}\gamma, \qquad (1)$$

where U_γ is a uniform probability distribution on $[A]^\gamma$ for all $\gamma \in [0,1]$.

Possibilistic variance

- The measure of f-weighted possibilistic variance of$A$ is the f-weighted average of the probabilistic variances of the respective uniform distributions on the level sets of A.

Possibilistic variance

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- The f-weighted *possibilistic variance* of $A \in \mathcal{F}$, with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0,1]$, is defined by

$$\mathrm{Var}_f(A) = \int_0^1 \mathrm{var}(U_\gamma) f(\gamma) \mathrm{d}\gamma = \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} f(\gamma) \mathrm{d}\gamma.$$

where U_γ is a uniform probability distribution on $[A]^\gamma$ for all $\gamma \in [0,1]$.

Possibilistic covariance

- The f-weighted possibilistic covariance between marginal possibility distributions of a joint possibility distribution is defined as the f-weighted average of probabilistic covariances between marginal probability distributions whose joint probability distribution is uniform on each level-set of the joint possibility distribution.

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- That is, the f-weighted *possibilistic covariance* between $A, B \in \mathcal{F}$, (with respect to their joint distribution C), can be written as

$$\mathrm{Cov}_f(A, B) = \int_0^1 \mathrm{cov}(X_\gamma, Y_\gamma) f(\gamma) d\gamma,$$

where X_γ and Y_γ are random variables whose joint distribution is uniform on $[C]^\gamma$ for all $\gamma \in [0, 1]$, and $\mathrm{cov}(X_\gamma, Y_\gamma)$ denotes their probabilistic covariance.

Possibilistic correlation

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- That is, the f-weighted measure of *possibilistic correlation* of $A, B \in \mathcal{F}$, (with respect to their joint distribution C), is defined as,

$$\rho_f^{\text{old}}(A,B) = \frac{\text{Cov}_f(A,B)}{\sqrt{\text{Var}_f(A)}\sqrt{\text{Var}_f(B)}}$$
$$= \frac{\int_0^1 \text{cov}(X_\gamma, Y_\gamma) f(\gamma) d\gamma}{\left(\int_0^1 \text{var}(U_\gamma) f(\gamma) d\gamma\right)^{1/2} \left(\int_0^1 \text{var}(V_\gamma) f(\gamma) d\gamma\right)^{1/2}} \qquad (2)$$

where V_γ is a uniform probability distribution on $[B]^\gamma$.

Possibilistic correlation

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Possibilistic correlation

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- After some computations we get $\rho_f^{old}(A, B) \approx 1.562$ for any weighting function f. We get here a value bigger than one since the variance of the first marginal distributions, X_γ, exceeds the variance of the uniform distribution on the same support.

Example with non-convex level set

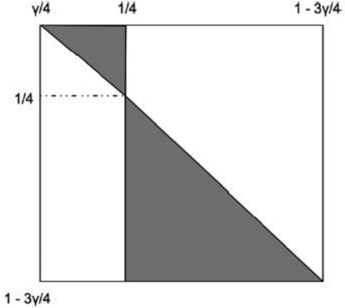

Figure: Example with non-convex level set

Possibilistic correlation

- We introduce a new index of interactivity between marginal distributions A and B of a joint possibility distribution C as the f-weighted average of the probabilistic correlation coefficients between the marginal probability distributions of a uniform probability distribution on $[C]^\gamma$ for all $\gamma \in [0, 1]$.

Possibilistic correlation

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Definition

The f-weighted index of interactivity of $A, B \in \mathcal{F}$ (with respect to their joint distribution C) is defined by

$$\rho_f(A,B) = \int_0^1 \rho(X_\gamma, Y_\gamma) f(\gamma) \mathrm{d}\gamma \tag{3}$$

where

$$\rho(X_\gamma, Y_\gamma) = \frac{\mathrm{cov}(X_\gamma, Y_\gamma)}{\sqrt{\mathrm{var}(X_\gamma)}\sqrt{\mathrm{var}(Y_\gamma)}}$$

and, where X_γ and Y_γ are random variables whose joint distribution is uniform on $[C]^\gamma$ for all $\gamma \in [0,1]$.

Possibilistic correlation

- It is clear that for any joint possibility distribution this new correlation coefficient always takes its value from interval $[-1, 1]$, since $\rho(X_\gamma, Y_\gamma) \in [-1, 1]$ for any $\gamma \in [0, 1]$ and $\int_0^1 f(\gamma) d\gamma = 1$.

Possibilistic correlation

- It is clear that for any joint possibility distribution this new correlation coefficient always takes its value from interval $[-1, 1]$, since $\rho(X_\gamma, Y_\gamma) \in [-1, 1]$ for any $\gamma \in [0, 1]$ and $\int_0^1 f(\gamma) d\gamma = 1$.
- Since $\rho_f(A, B)$ measures an average index of interactivity between the level sets of A and B, we will call this measure as the f-weighted possibilistic correlation coefficient.

Example 1.

- Consider the case, when $A(x) = B(x) = (1 - x) \cdot \chi_{[0,1]}(x)$, for $x \in \mathbb{R}$, that is $[A]^\gamma = [B]^\gamma = [0, 1 - \gamma]$, for $\gamma \in [0, 1]$.

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- Suppose that their joint possibility distribution is given by $F(x, y) = (1 - x - y) \cdot \chi_T(x, y)$, where

$$T = \left\{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1\right\}.$$

A γ-level set of F is computed by

$$[F]^\gamma = \left\{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1 - \gamma\right\}.$$

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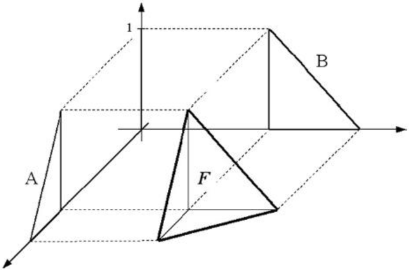

Figure: Illustration of joint possibility distribution F.

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- The f-weighted possibilistic correlation of A and B:

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- We note here that using the former definition (2) we would obtain $\rho_f^{\text{old}}(A, B) = -1/3$ for the correlation coefficient.

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- So $\mathrm{Cov}_f(A, B) = 0$ and $\rho_f(A, B) = 0$ for any weighting function f.

Perfect correlation

- Fuzzy numbers A and B are said to be in perfect correlation, if there exist $q, r \in \mathbb{R}$, $q \neq 0$ such that their joint possibility distribution is defined by

$$C(x_1, x_2) = A(x_1) \cdot \chi_{\{qx_1 + r = x_2\}}(x_1, x_2) = B(x_2) \cdot \chi_{\{qx_1 + r = x_2\}}(x_1, x_2), \tag{4}$$

where $\chi_{\{qx_1+r=x_2\}}$, stands for the characteristic function of the line

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- In this case we have

$$[C]^\gamma = \left\{(x, qx + r) \in \mathbb{R}^2 \middle| x = (1-t)a_1(\gamma) + ta_2(\gamma), t \in [0, 1]\right\}$$

where $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$; and $[B]^\gamma = q[A]^\gamma + r$, for any $\gamma \in [0, 1]$, and, finally,

$$B(x) = A\left(\frac{x - r}{q}\right),$$

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$$B(x) = A\left(\frac{x - r}{q}\right),$$

  for all $x \in \mathbb{R}$.
- Furthermore, A and B are in a perfect positive (negative) correlation if q is positive (negative) in (4).

Perfect correlation

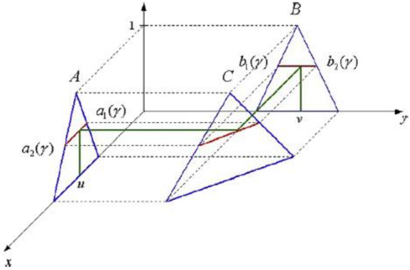

Figure: Perfect negative correlation.

Perfect correlation

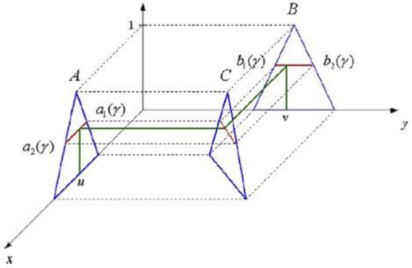

Figure: Perfect positive correlation.

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  for $x \in \mathbb{R}$, that is $[A]^\gamma = [B]^\gamma = [0, 1 - \gamma]$, for $\gamma \in [0, 1]$. Suppose that their joint possibility distribution is given by 'pure shares' (the marginal distributions themselves),
  $$C(x,y) = (1 - x - y) \cdot \chi_T(x,y),$$
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Theorem

If A and B are fuzzy numbers and their joint possibility distribution has the following form:

$$C(x,y) = \begin{cases} A(x) & \text{if } y = 0 \\ B(y) & \text{if } x = 0 \\ 0 & \text{otherwise,} \end{cases}$$

then

$$|\ \rho_f(A,B)\ | \leq \frac{3}{5}.$$

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Example

Suppose we have two probability distributions, X and Y, with two-dimensional standard normal joint distribution, and the correlation coefficient of X and Y is r. Then the relationship between X and Y^2 is clearly not linear, their correlation coefficient is 0. But if r is close to 1, the relationship between X and Y^2 is still very strong. And in this case the correlation ratio takes the value r^2.

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- The correlation ratio is invariant to multiplicative changes in the first argument:

$$\eta^2(kX|Y) = \eta^2(X|Y)$$

Basic properties II.

- If $E(X|Y = y)$ is linear function of y (i.e. there is a linear relationship between random variables $E(X|Y)$ and Y) this will give the same result as the square of the correlation coefficient ($\rho(X, Y)$), otherwise the correlation ratio will be larger in magnitude:

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Theorem

$$\eta^2(X|Y) = \sup_f \rho^2(X, f(Y)),$$

where we take the supremum for all the functions f, such that $f(Y)$ has finite variance. The correlation can reach its maximum if $f(y) = aE(X|Y=y) + b$.

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- The difference between $\eta^2(X|Y)$ and $\rho^2(X,Y)$ can be interpreted as the degree of non-linearity between X and Y

$$\eta^2(X|Y) - \rho^2(X,Y) = \frac{1}{D^2(X)} \left\{ \min_{a,b} E(Y-(aX+b))^2 - \min_f E(Y-f(X))^2 \right\}$$

Correlation ratio

Definition

Let us denote A and B the marginal possibility distributions of a given joint possibility distribution C. Then the *f-weighted possibilistic correlation ratio* of marginal possibility distribution A with respect to marginal possibility distribution B is defined by

$$\eta_f^2(A|B) = \int_0^1 \eta^2(X_\gamma|Y_\gamma) f(\gamma) \mathrm{d}\gamma \tag{5}$$

where X_γ and Y_γ are random variables whose joint distribution is uniform on $[C]^\gamma$ for all $\gamma \in [0, 1]$, and $\eta^2(X_\gamma|Y_\gamma)$ denotes their probabilistic correlation ratio.

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- If A and B are symmetrical fuzzy numbers, then

$$\eta^2(A|B) = \eta^2(B|A) = 0.$$

Linear Relationship

Consider the case, when

$$A(x) = B(x) = (1 - x) \cdot \chi_{[0,1]}(x),$$

for $x \in \mathbb{R}$, that is $[A]^\gamma = [B]^\gamma = [0, 1-\gamma]$, for $\gamma \in [0,1]$. Suppose that their joint possibility distribution is given by $C(x,y) = (1 - x - y) \cdot \chi_T(x,y)$, where

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Linear Relationship

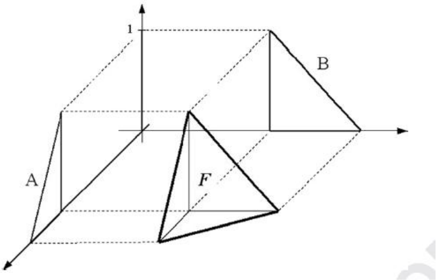

Figure: Illustration of joint possibility distribution C.

Linear Relationship

- $$\begin{aligned} D^2[E(X_\gamma|Y_\gamma)] &= E(E(X_\gamma|y) - E(X_\gamma))^2 \\ &= \int_0^{1-\gamma} (\frac{1-\gamma-y}{2} - \frac{1-\gamma}{3})^2 \frac{2(1-\gamma-y)}{(1-\gamma)^2} \mathrm{d}y \\ &= \frac{(1-\gamma)^2}{72}. \end{aligned}$$

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- From this the f-weighted possibilistic correlation ratio of A with respect to B is,
$$\eta_f^2(A|B) = \int_0^1 \frac{1}{4} f(\gamma) \mathrm{d}\gamma = \frac{1}{4}.$$

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$$E(X_\gamma|Y_\gamma = y) = \frac{1-\gamma-y}{2}$$
$$= \frac{1-\gamma}{3} - \frac{1}{2}\left(y - \frac{1-\gamma}{3}\right) = E(X_\gamma) - \rho_f(X_\gamma, Y_\gamma)(y - E(Y_\gamma)).$$

A Ball-Shaped Joint Distribution

- $$A(x) = B(x) = (1 - x^2) \cdot \chi_{[0,1]}(x),$$

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- The joint possibility distribution is ball-shaped, that is,
  $$C(x,y) = (1 - x^2 - y^2) \cdot \chi_T(x,y),$$
  where
  $$T = \left\{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 1\right\}.$$
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- The f-weighted possibilistic correlation ratio of B with respect A is
  $$\eta_f^2(B|A) = \int_0^1 \frac{27\pi^2 - 256}{36\pi^2 - 256} f(\gamma)\mathrm{d}\gamma = \frac{27\pi^2 - 256}{36\pi^2 - 256}.$$

Different marginals 1.

- The marginal distributions are

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- The f-weighted possibilistic correlation ratio of B with respect to A is

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- The f-weighted possibilistic correlation ratio of B with respect to A is
$$\eta_f^2(A|B) = \int_0^1 \frac{3}{19} f(\gamma) \mathrm{d}\gamma = \frac{3}{19} \neq \eta_f^2(B|A).$$

Different marginals 2.

- The marginal distributions are

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- We will examine the relationship/difference between $\eta_f^2(A|B)$ and $\rho_f(A, B)$ in different cases (C has convex γ-cuts)

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- Use it in applications for sensitivity analysis: if **A** is a set of fuzzy numbers $A_1, \ldots, A_n$, then $\eta_f^2(G(\mathbf{A})|A_i)$ represents the fraction of the variance of $G(\mathbf{A})$ which is "explained" by A_i .

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A *quasi fuzzy number* A is a fuzzy set of the real line with a normal, fuzzy convex and continuous membership function satisfying the limit conditions [?]

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A *quasi triangular* fuzzy number is a quasi fuzzy number with a unique maximizing point.If A is a quasi fuzzy number, then $[A]^\gamma$ is a closed convex (compact) subset of $\mathbb{R}$ for any $\gamma > 0$.

Example

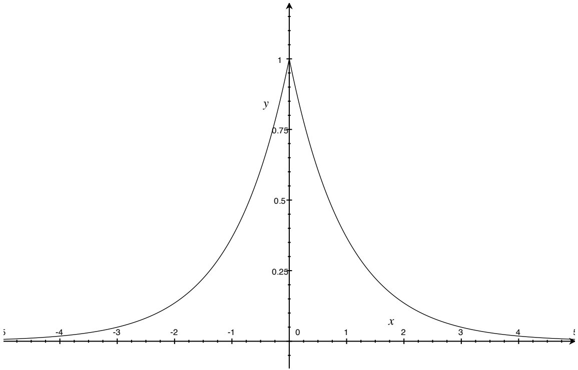

Figure: A quasi triangular fuzzy number with membership function $e^{-|x|}$.

Possibilistic mean

- We can extend the definition of possibilistic mean to quasy fuzzy numbers

Definition

The f-weighted *possibilistic mean value* of $A \in \mathcal{Q}$ is defined as

$$E_f(A) = \int_0^1 E(U_\gamma) f(\gamma) d\gamma = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} f(\gamma) d\gamma,$$

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- However, for a symmetric quasi fuzzy number A we get $E_f(A) = a$, where a is the center of symmetry, for any weighting function f.

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Example

Consider the following quasi fuzzy number

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \dfrac{1}{\sqrt{x+1}} & \text{if } 0 \leq x \end{cases}$$

In this case $a_1(\gamma) = 0, \quad a_2(\gamma) = \dfrac{1}{\gamma^2} - 1$, and its possibilistic mean value can not be computed, since the following integral does not exist (not finite),

$$E(A) = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma \mathrm{d}\gamma = \int_0^1 \left(\frac{1}{\gamma^2} - 1 \right) \gamma \mathrm{d}\gamma = \int_0^1 \left(\frac{1}{\gamma} - \gamma \right) \mathrm{d}\gamma.$$

Possibilistic mean

- The main result

Theorem

If A is a non-symmetric quasi fuzzy number then $E_f(A)$ is finite (where the weighting function is $f(\gamma) = 2\gamma$) if and only if there exist real numbers $\varepsilon, \delta > 0$, such that,

$$\mu_A(x) = O\left(x^{-\frac{1}{2}-\varepsilon}\right),$$

if $x \to +\infty$ and

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- If we consider other weighting functions, we need to require that $\mu_A(x) = O(x^{-1-\varepsilon})$, when $x \to +\infty$ (in the worst case, when $f(\gamma) = 1$, $\frac{1}{\gamma}$ is the critical growth rate.)

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Consider the following quasi triangular fuzzy number,

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \dfrac{1}{x+1} & \text{if } 1 \le x \end{cases}$$

In this case we have,

$$a_1(\gamma) = 0, \quad a_2(\gamma) = \frac{1}{\gamma} - 1,$$

and its possibilistic mean value is,

$$E(A) = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma \mathrm{d}\gamma = \int_0^1 \left(\frac{1}{\gamma} - 1\right)\gamma \mathrm{d}\gamma = \int_0^1 (1-\gamma)\mathrm{d}\gamma = 1/2.$$

Example

- This example is very important since the volume of A can not be normalized since $\int_0^\infty \mu_A(x)\mathrm{d}x$ does not exist. In other words, μ_A can not be considered as a density function of any random variable.

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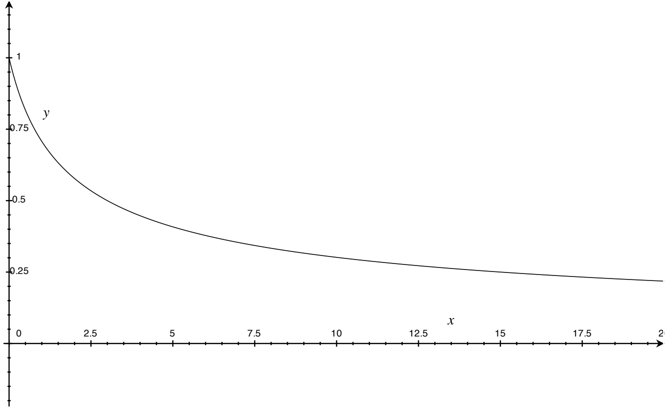

Figure: Quasi triangular fuzzy number $1/(x+1), x \geq 0$.

Possibilistic variance

Definition

The measure of f-weighted possibilistic variance of a quasi fuzzy number A is the f-weighted average of the probabilistic variances of the respective uniform distributions on the level sets of A. That is, the f-weighted *possibilistic variance* of A is defined as the covariance of A with itself

$$\mathrm{Var}_f(A) = \int_0^1 \mathrm{var}(U_\gamma) f(\gamma) \mathrm{d}\gamma = \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} f(\gamma) \mathrm{d}\gamma.$$

where U_γ is a uniform probability distribution on $[A]^\gamma$ for all $\gamma > 0$. The value of $\mathrm{Var}_f(A)$ does not depend on the boundedness of the support of A. If $f(\gamma) = 2\gamma$ then we simple write $\mathrm{Var}(A)$.

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Probabilistic fuzzy numbers

Theorem

If A is a probabilistic (quasi) fuzzy number (it means that $A(x)$ is the membership function of a fuzzy number and also the density function of a probability distribution) with center a then

$$\mid E_f(A) - a \mid \leq \mid M(A) - a \mid$$

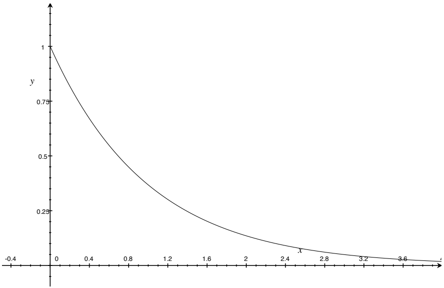

Figure: Quasi triangular fuzzy number and density function of an exponential random variable with parameter one: e^{-x}, $x \geq 0$.

Real option

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- Contrasted with more standard techniques of capital budgeting (such as NPV), where only the most likely or representative outcomes are modelled.
- Uncertainty inherent in investment projects is usually accounted for by risk-adjusting probabilities.

Net present value

The total present value of a time series of cash flows. It is a standard method for using the time value of money to appraise long-term projects. Each cash inflow/outflow is discounted back to its present value, then they are summed.

$$NPV = \sum_{t=0}^{T} \frac{C_t}{(1+i)^t},$$

where:
t - the time of the cash flow
i - the discount rate
C_t - the net cash flow at time t

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Option Pricing Models

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- Partial differential equation
- The Black-Scholes Model:limiting case of the binomial.

The formula

Value of the call=$SN(d_1) - Ke^{-rt}N(d_2)$,
where:

$$d_1 = \frac{ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}},$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

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- This may hold for some efficiently traded financial securities, but may not hold for real investments that do not have existing markets
- Practical use of real options can only be reached through usable and understandable methods that use the types of inputs that companies are already collecting and generating

Definition

A fuzzy set A is called triangular fuzzy number with peak (or center) a, left width $\alpha > 0$ and right width $\beta > 0$ if its membership function has the following form

$$A(t) = \begin{cases} 1 - \dfrac{a-t}{\alpha} & \text{if } a-\alpha \le t \le a \\ 1 - \dfrac{t-a}{\beta} & \text{if } a \le t \le a+\beta \\ 0 & \text{otherwise} \end{cases}$$

and we use the notation $A = (a, \alpha, \beta)$. It can easily be verified that

$$[A]^\gamma = [a - (1-\gamma)\alpha, a + (1-\gamma)\beta], \ \forall \gamma \in [0, 1].$$

The support of A is $(a - \alpha, b + \beta)$.

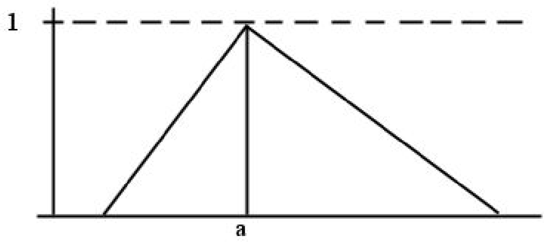

Figure: A triangular fuzzy number A, defined by three points $\{a, \alpha, \beta\}$.

Definition

A fuzzy set A is called trapezoidal fuzzy number with tolerance interval $[a, b]$, left width α and right width β if its membership function has the following form

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and we use the notation

$$A = (a, b, \alpha, \beta). \tag{6}$$

Why we use fuzzy numbers

- To estimate future cash flows and discount rates we usually employ educated guesses, based on expected values or other statistical techniques, which is consistent with the use of fuzzy numbers.

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- These models are more in line with reality, as they do not simplify uncertain distribution-like observations to a single point estimate that conveys the sensation of no-uncertainty.
- The most used fuzzy numbers are trapezoidal and triangular fuzzy numbers, because they make many operations possible and are intuitively understandable and interpretable.

The Datar-Mathews method uses a simulation to generate a probability distribution of project outcomes from project cash-flow scenarios given by the responsible project managers - then the probability weighted mean value of the positive outcomes is calculated and multiplied by the probability of the positive outcomes (%) over all of the outcomes (100%). The answer is real option value. The Datar-Mathews method is shown to correspond to the answer from the Black-Scholes model when the same constraints are used.

The method is based on simulation generated probability distributions for the NPV of future project outcomes. The project outcome probability distributions are used to generate a pay-off distribution, where the negative outcomes (subject to terminating the project) are truncated into one chunk that will cause a zero pay-off, and where the probability weighted average value of the resulting pay-off distribution is the real option value. The DMM shows that the real-option value can be understood as the probability-weighted average of the pay-off distribution.

Fuzzy Pay-Off Method

Fuzzy Pay-Off Method for Real Option Valuation is a new method for valuing real options, created in 2008. It is based on the use of fuzzy logic and fuzzy numbers for the creation of the pay-off distribution of a possible project (real option). The structure of the method is similar to the probability theory based Datar-Mathews method.

The main observations

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- The fuzzy NPV of a project is (equal to) the pay-off distribution of a project value that is calculated with fuzzy numbers
- The mean value of the positive values of the fuzzy NPV is the possibilistic mean value of the positive fuzzy NPV values.
- Real option value calculated from the fuzzy NPV is the possibilistic mean value of the positive fuzzy NPV values multiplied with the positive area of the fuzzy NPV over the total area of the fuzzy NPV.

In other words, the real option value can be derived (without any simulation whatsoever) from the fuzzy NPV. These are the blocks that together make the fuzzy pay-off method for real option valuation.

Definition

We calculate the real option value from the fuzzy NPV as follows

$$\mathrm{ROV} = \frac{\int_0^\infty A(x)\mathrm{d}x}{\int_{-\infty}^\infty A(x)\mathrm{d}x} \times E(A_+) \tag{7}$$

where A stands for the fuzzy NPV, $E(A_+)$ denotes the fuzzy mean value of the positive side of the NPV, and $\int_{-\infty}^\infty A(x)\mathrm{d}x$ computes the area below the whole fuzzy number A, and $\int_0^\infty A(x)\mathrm{d}x$ computes the area below the positive part of A.

The triangular case

The membership function of the right-hand side of a triangular fuzzy number truncated at point $a - \alpha + z$, where $0 \leq z \leq \alpha$:

$$(A|z)(t) = \begin{cases} 0 & \text{if } t \leq a - \alpha + z \\ A(t) & \text{otherwise} \end{cases}$$

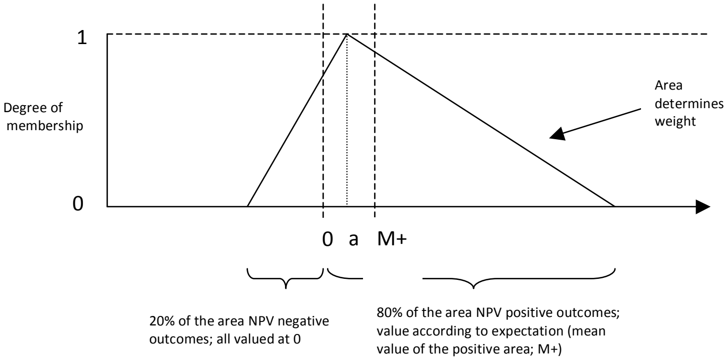

Figure: A triangular fuzzy number A, defined by three points $\{a, \alpha, \beta\}$ describing the NPV of a prospective project; (percentages 20% and 80% are for illustration purposes only).

The triangular case

$$E(A|z) = I_1 + I_2 = \int_0^{z_1} \gamma(a - \alpha + z + a + (1-\gamma)\beta)d\gamma + \int_{z_1}^1 \gamma(a - (1-\gamma)\alpha + a + (1-\gamma)\beta)d\gamma \quad (8)$$

where

$$z_1 = 1 - \frac{\alpha - z}{\alpha} = \frac{z}{\alpha}.$$

The triangular case

$$E(A|z) = \frac{z^3}{6\alpha^2} + a + \frac{\beta - \alpha}{6}.$$

If $z = \alpha - a$ then $A|z$ becomes A_+, the positive side of A, and therefore, we get

$$E(A_+) = \frac{(\alpha - a)^3}{6\alpha^2} + a + \frac{\beta - \alpha}{6}.$$

The triangular case

The membership function of the right-hand side of a triangular fuzzy number truncated at point $a-\alpha+z$, where $\alpha \leq z \leq \alpha+\beta$:

$$E(A_{+}) = \frac{(\beta - z + \alpha)^2}{6\beta^2}(6a - 5\alpha + 5z + \beta).$$

The triangular case

To compute the real option value with the above formulas we must calculate the ratio between the positive area of the triangular fuzzy number and the total area of the same number and multiply this by $E(A_+)$, the fuzzy mean value of the positive part of the fuzzy number A, according to the formula (7).

The trapezoidal case

For computing the real option value from an NPV (pay-off) distribution of a trapezoidal form we must consider a trapezoidal fuzzy pay-off distribution A defined by

$$A(u) = \begin{cases} \dfrac{u}{\alpha} - \dfrac{a_1 - \alpha}{\alpha} & \text{if } a_1 - \alpha \leq u \leq a_1 \\ 1 & \text{if } a_1 \leq u \leq a_2 \\ \dfrac{u}{-\beta} + \dfrac{a_2 + \beta}{\beta} & \text{if } a_2 \leq u \leq a_2 + \beta \\ 0 & \text{otherwise} \end{cases}$$

The trapezoidal case

The γ-level of A is defined by $[A]^\gamma = [\gamma\alpha + a_1 - \alpha, -\gamma\beta + a_2 + \beta]$ and its expected value is caculated by

$$E(A) = \frac{a_1 + a_2}{2} + \frac{\beta - \alpha}{6}.$$

The trapezoidal case

Then we have the following five cases, considering the position of 0 in the fuzzy NPV:

Case 1: $z < a_1 - \alpha$. In this case we have $E(A|z) = E(A)$.

The trapezoidal case

Case 2: $a_1 - \alpha < z < a_1$. Then introducing the notation,

$$\gamma_z = \frac{z}{\alpha} - \frac{a_1 - \alpha}{\alpha}$$

we find

$$[A]^\gamma = \begin{cases} (z, -\gamma\beta + a_2 + \beta) & \text{if } \gamma \leq \gamma_z \\ (\gamma\alpha + a_1 - \alpha, -\gamma\beta + a_2 + \beta) & \text{if } \gamma_z \leq \gamma \leq 1 \end{cases}$$

and,

$$E(A|z) = \int_0^{\gamma_z} \gamma(z - \gamma\beta + a_2 + \beta)d\gamma + \int_{\gamma_z}^1 \gamma(\gamma\alpha + a_1 - \alpha - \gamma\beta + a_2 + \beta)d\gamma$$

$$= \frac{a_1 + a_2}{2} + \frac{\beta - \alpha}{6} + (z - a_1 + \alpha)\frac{\gamma_z^2}{2} - \alpha\frac{\gamma_z^3}{3}$$

The trapezoidal case

Case 3: $a_1 < z < a_2$. In this case $\gamma_z = 1$ and

$$[A]^\gamma = [z, -\gamma\beta + a_2 + \beta]$$

and we get,

$$E(A|z) = \int_0^1 \gamma(z - \gamma\beta + a_2 + \beta)d\gamma = \frac{z + a_2}{2} + \frac{\beta}{6}$$

The trapezoidal case

Case 4: $a_2 < z < a_2 + \beta$. In this case we have

$$\gamma_z = \frac{z}{-\beta} + c\frac{a_2 + \beta}{\beta}$$

and,

$$[A]^\gamma = [z, -\gamma\beta + a_2 + \beta],$$

if $\gamma < \gamma_z$ and we find,

$$E(A|z) = \int_0^{\gamma_z} \gamma(z - \gamma\beta + a_2 + \beta)d\gamma = (z + a_2 + \beta)\frac{\gamma_z^2}{2} - \beta\frac{\gamma_z^3}{3}.$$

Case 5: $a_2 + \beta < z$. Then it is easy to see that $E(A|z) = 0$

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- The simplicity of the presented method over more complex methods.
- Using triangular and trapezoidal fuzzy numbers make very easy implementations possible with the most commonly used spreadsheet software; this opens avenues for real option valuation to find its way to more practitioners.
- The method is flexible as it can be used when the fuzzy NPV is generated from scenarios or as fuzzy numbers from the beginning of the analysis.

Advantages of the method

As information changes, and uncertainty is reduced, this should be reflected in the fuzzy NPV, the more there is uncertainty the wider the distribution should be, and when uncertainty is reduced the width of the distribution should decrease. Only under full certainty should the distribution be represented by a single number, as the method uses fuzzy NPV there is a possibility to have the size of the distribution decrease with a lesser degree of uncertainty, this is an advantage over probability based methods.

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Thank you for your attention!