An interactive approach to solve multicriteria median location problem

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Outline

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Interactive process

The basic steps in interactive algorithms

- find an initial feasible solution,
- interact with the decision maker, and
- obtain a new solution (or a set of new solutions). If the new solution (or one of them) or one of the previous solutions is acceptable to the decision maker, stop. Otherwise, go to the previous step.

Two major requirements are set for a scalarizing function
Y. Sawaragi, H. Nakayama and T. Tanino, Theory of multiobjective optimization, Academic Press, Orlando, 1985

- correctness: every solution found by means of scalarization should be (weakly) Pareto optimal, and
- completeness: it should be able to cover the entire set of Pareto optimal solutions.

$$f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$$

$$x \in X$$

$$\min_{x \in X} f_i(x), \quad i \in N_m = \{1, 2, \ldots, m\}.$$

A set of minima of the i'th objective function

$$M^i(X) = arg \min_{x \in X} f_i(x), \quad i \in N_m$$

If

$$\bigcap_{i=1}^{m} M^i(X) \neq \emptyset,$$

then there exists at least one solution which delivers a minimum for all objectives. Such a solution can be called an *ideal solution*.

Pareto optimality $P^m(X)$

A decision vector $x^* \in X$ is Pareto optimal if there exists no $x \in X$ such that $f_i(x) \leq f_i(x^*)$ for all $i \in N_m$ and $f_j(x) < f_j(x^*)$ for at least one index j.

Slater optimality $Sl^m(X)$

A decision vector $x^* \in X$ is weakly Pareto optimal if there exists no $x \in X$ such that $f_i(x) < f_i(x^*)$ for all $i \in N_m$.

Ideal and *nadir* objective vectors $f^I = (f_1^I, \ldots, f_m^I)$ and $f^N = (f_1^N, \ldots, f_m^N)$

$$f_i^I = \min_{x \in P^m(X)} f_i(x), \quad i \in N_m,$$

$$f_i^N = \max_{x \in P^m(X)} f_i(x), \quad i \in N_m.$$

Achievement scalarizing functions

A. P. Wierzbicki, The use of reference objectives in multiobjective optimization. Lecture notes in economics and mathematical systems, 177 (1980)

The scalarized problem

$$s_R : \mathbb{R}^m \to \mathbb{R} \qquad \min_{x \in X} s_R(f(x)) \tag{1}$$

An ASF $s_R : \mathbb{R}^m \to \mathbb{R}$ is said to be

1. Increasing, if for any $y^1, y^2 \in \mathbb{R}^m$, $y_i^1 \leq y_i^2$ for all $i \in N_m$, then $s_R(y^1) \leq s_R(y^2)$.
2. Strictly increasing, if for any $y^1, y^2 \in \mathbb{R}^m$, $y_i^1 < y_i^2$ for all $i \in N_m$, then $s_R(y^1) < s_R(y^2)$.
3. Strongly increasing, if for any $y^1, y^2 \in \mathbb{R}^m$, $y_i^1 \leq y_i^2$ for all $i \in N_m$ and $y^1 \neq y^2$, then $s_R(y^1) < s_R(y^2)$.

Optimality conditions for ASFs

Theorem

1. *Let s_R be strongly (strictly) increasing. If $x^* \in X$ is an optimal solution of problem* (1), *then x^* is (weakly) Pareto optimal.*
2. *If s_R is increasing and the solution of* (1) *$x^* \in X$ is unique, then x^* is Pareto optimal.*

Theorem

If s_R is strictly increasing and $x^* \in X$ is weakly Pareto optimal, then it is a solution of (1) *with $f^R = f(x^*)$ and the optimal value of s_R is zero.*

A parameterized ASF

$$s_R^q(f(x),\lambda) = \max_{I^q \subseteq N_m : |I^q| = q} \left\{ \sum_{i \in I^q} \max\left[\lambda_i (f_i(x) - f_i^R), 0\right] \right\},$$

where $q \in N_m$ and $\lambda = \{\lambda_1, \ldots, \lambda_m\}, \lambda_i > 0, i \in N_m$.

- for $q \in N_m : s_R^q(f(x),\lambda) \geq 0$;
- $q = 1 : s_R^1(f(x),\lambda) = \max_{i \in N_m} \max[\lambda_i (f_i(x) - f_i^R), 0] \cong s_R^\infty(f(x),\lambda)$;
- $q = m : s_R^m(f(x),\lambda) = \sum_{i \in N_m} \max[\lambda_i (f_i(x) - f_i^R), 0] = s_R^1(f(x),\lambda)$.

$$\min_{x \in X} s_R^q(f(x),\lambda). \tag{2}$$

Yu. Nikulin, K. Miettinen and M. M. Mäkelä, A new achievement scalarizing function based on parameterization in multiobjective optimization, OR Spectrum, 34 (1) (2012)

Theorem

Given problem (2)*, let* f^R *be a reference point such that there exists no feasible solution whose image strictly dominates* f^R*. Also assume* $\lambda_i > 0$ *for all* $i \in N_m$*. Then among the optimal solutions of problem* (2) *is a weakly Pareto optimal solution.*

Theorem

Given problem (2)*, let* f^R *be a reference point. Also assume* $\lambda_i > 0$ *for all* $i \in N_m$*. Then among the optimal solutions of problem* (2) *there exists at least one Pareto optimal solution.*

Classification of the objective functions

- f_i values are desired to be improved (i.e. decreased),
- f_i values may be impaired (i.e. increased).

The case of three objectives $m = 3$

$$s_R^q(f(x), \lambda) = \max_{I^q \subseteq \{1,2,3\}:|I^q|=q} \left\{ \sum_{i \in I^q} \max\left[\lambda_i (f_i(x) - f_i^R), 0\right] \right\},$$

where $q = 1, 2, 3$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3),\ \lambda_i > 0,\ i \in N_3$.

$f^R = f^I = (f_1^I, f_2^I, f_3^I)$

$q = 1$

$$s_R^1(f(x), \lambda) = \max\Big\{\max[\lambda_1(f_1(x) - f_1^R), 0], \max[\lambda_2(f_2(x) - f_2^R), 0], \max[\lambda_3(f_3(x) - f_3^R), 0]\Big\}$$

$$= \max\Big\{\lambda_1(f_1(x) - f_1^I), \lambda_2(f_2(x) - f_2^I), \lambda_3(f_3(x) - f_3^I)\Big\};$$

$q = 2$

$$s_R^2(f(x),\lambda) = \max\Big\{\max[\lambda_1(f_1(x) - f_1^R),0] + \max[\lambda_2(f_2(x) - f_2^R),0],$$
$$\max[\lambda_1(f_1(x) - f_1^R),0] + \max[\lambda_3(f_3(x) - f_3^R),0],$$
$$\max[\lambda_2(f_2(x) - f_2^R),0] + \max[\lambda_3(f_3(x) - f_3^R),0]\Big\}$$
$$= \max\Big\{\lambda_1(f_1(x) - f_1^I) + \lambda_2(f_2(x) - f_2^I),$$
$$\lambda_1(f_1(x) - f_1^I) + \lambda_3(f_3(x) - f_3^I),$$
$$\lambda_2(f_2(x) - f_2^I) + \lambda_3(f_3(x) - f_3^I)\Big\};$$

$q = 3$

$$s_R^3(f(x), \lambda) = \max\Big\{\max[\lambda_1(f_1(x) - f_1^R), 0] + \max[\lambda_2(f_2(x) - f_2^R), 0] + \max[\lambda_3(f_3(x) - f_3^R), 0]\Big\}$$

$$= \lambda_1(f_1(x) - f_1^I) + \lambda_2(f_2(x) - f_2^I) + \lambda_3(f_3(x) - f_3^I);$$

Corner coordinates $q = 1$

$(\alpha/\lambda_1 + f_1^I, \alpha/\lambda_2 + f_2^I, \alpha/\lambda_3 + f_3^I)$

Top vertex coordinates $q = 2$

$(\alpha/2\lambda_1 + f_1^I, \alpha/2\lambda_2 + f_2^I, \alpha/2\lambda_3 + f_3^I)$

Normal vector $q = 3$

$(\lambda_1, \lambda_2, \lambda_3)$

Problem parameters

A set of sectors that should be evacuated

$$S,\ |S| = n, n \in \mathbb{N}$$

Each region $i \in S, i = 1, \ldots, n$ has a_i habitants
The number of candidate shelters

$$E, |E| = l, l \in \mathbb{N}$$

The number of shelters to be located

$$p \leq E$$

Path length from a sector i to a shelter j

$$d_{ij} \in \mathbb{R},\ j = 1, \ldots, l$$

Problem parameters

Risk associated with a path from a sector i to a shelter j

$$r_{ij} \in (0, 1)$$

Risk associated with a shelter j

$$r_j \in (0, 1)$$

Capacity (number of individuals) allowed in a jth candidate shelter

$$K_j \in \mathbb{N}$$

Minimum number of individuals required for opening a jth shelter

$$k_j \in \mathbb{N}$$

Multiobjective p-median problem

The distance required for the population to reach its shelter

$$\min \sum_{i=1}^{n} \sum_{j=1}^{l} a_i d_{ij} x_{ij}$$

The risk faced by the population as it travels to its shelter

$$\min \sum_{i=1}^{n} \sum_{j=1}^{l} a_i r_{ij} x_{ij}$$

Total risks associated with staying in the shelter

$$\min \sum_{i=1}^{n} \sum_{j=1}^{l} a_i r_j x_{ij}$$

Constraints

$$\sum_{j=1}^{l} x_{ij} = 1, \quad i = 1, \dots, n$$

(one evacuation path is chosen for each sector, with n the number of sectors)

$$\sum_{i=1}^{n} a_i x_{ij} \leq K_j y_j, \quad j = 1, \dots, l$$

(the maximum capacity for shelter j is not exceeded, with l the total number of candidate shelters)

$$\sum_{i=1}^{n} a_i x_{ij} \geq k_j y_j, \quad j = 1, \dots, l$$

(the minimum number of individuals required to open shelter j before it is opened)

Constraints

$$\sum_{j=1}^{l} y_j = p, \quad j = 1, \ldots, l$$

(ensures p of the l candidate shelters are opened)

$$x_{ij} \in \{0, 1\}, \quad i = 1, \ldots, n, \ j = 1, \ldots, l$$

$$y_j \in \{0, 1\}, \quad j = 1, \ldots, l$$

Data set

Distance matrix

$$(d_{ij}) = \begin{pmatrix} 18 & 29 & 0 & 21 & 27 \\ 17 & 21 & 14 & 21 & 30 \\ 24 & 26 & 27 & 30 & 19 \\ 22 & 27 & 18 & 16 & 17 \\ 13 & 13 & 26 & 15 & 7 \end{pmatrix},$$

here length $d_{ij} = 0$ means that there is no path from a sector i to a shelter j;

Number of individuals in each sector

$$a = (5,\ 18,\ 21,\ 19,\ 29);$$

Data set

Risk associated with path from a sector i to a shelter j

$$(r_{ij}) = \begin{pmatrix} 0.7000 & 0.0119 & 0.9975 & 0.7582 & 0.9022 \\ 0.7745 & 0.2775 & 0.4996 & 0.3107 & 0.3198 \\ 0.6965 & 0.4296 & 0.4775 & 0.0606 & 0.6897 \\ 0.7466 & 0.4359 & 0.2137 & 0.1030 & 0.3448 \\ 0.6685 & 0.0985 & 0.8209 & 0.6778 & 0.0190 \end{pmatrix};$$

Risk associated with a shelter j

$$r = (0.2936,\ 0.1979,\ 0.0786,\ 0.9210,\ 0.5971);$$

Capacity of a shelter j

$$K = (26,\ 25,\ 65,\ 40,\ 47);$$

Minimum number of individuals required for opening shelter j

$$k = (9,\ 8,\ 6,\ 9,\ 9).$$

$$f^l = (1353,\ 10.9402,\ 23.9437)$$

$$\lambda_1 = 1/f^l_1,\ \lambda_2 = 1/f^l_2,\ \lambda_3 = 1/f^l_3$$

$$\lambda = (0.000739098,\ 0.0914061,\ 0.0417647)$$

A step of interactive process

$$\lambda_i^c = \begin{cases} \lambda_i^{c-1} + 0.5\lambda_i^{c-1}/(c-1) & \text{if } f_i \text{ is desired to be decreased} \\ \lambda_i^{c-1} - 0.5\lambda_i^{c-1}/(c-1) & \text{if } f_i \text{ is desired to be increased} \end{cases}$$

$$i = 1, 2, 3,\ c \in \mathbb{N}$$

The most preferred solution

$$\lambda = (0.01,\ 0.0001,\ 20)$$

$$\text{MPS} = (1487,\ 31.0761,\ 29.7186)$$

Results for q=1

iteration	lambda	mean value	test
1	"(0.0007,0.0914,0.0417)"	"(1597,17.5895,41.0176)"	111.398
2	"(0.0011,0.0457,0.0626)"	"(1711,29.8182,29.0974)"	224.004
3	"(0.0013,0.0342,0.0469)"	"(1572,24.2920,29.5272)"	85.2705
4	"(0.0016,0.0285,0.0391)"	"(1572,24.2920,29.5272)"	85.2705
5	"(0.0018,0.0249,0.0342)"	"(1572,24.2920,29.5272)"	85.2705
6	"(0.002,0.0224,0.0308)"	"(1597,17.5895,41.0176)"	111.398
7	"(0.0021,0.0206,0.0334)"	"(1487,31.0761,29.7186)"	0

Results for q=2

iteration	lambda	mean value	test
1	"(0.0007,0.0914,0.0417)"	"(1698,10.9401,42.7022)"	212.356
2	"(0.0011,0.0457,0.0626)"	"(1572,24.2920,29.5272)"	85.2705
3	"(0.0013,0.0342,0.0469)"	"(1487,31.0761,29.7186)"	0

Results for q=3

iteration	lambda	mean value	test
1	"(0.0007,0.0914,0.0417)"	"(1698,10.9401,42.7022)"	212.356
2	"(0.0011,0.0457,0.0626)"	"(1572,24.2920,29.5272)"	85.2705
3	"(0.0013,0.0342,0.0469)"	"(1572,24.2920,29.5272)"	85.2705
4	"(0.0016,0.0285,0.0391)"	"(1572,24.2920,29.5272)"	85.2705
5	"(0.0018,0.0249,0.0342)"	"(1572,24.2920,29.5272)"	85.2705
6	"(0.0020,0.0224,0.0308)"	"(1517,27.7326,30.0057)"	30.1871
7	"(0.0021,0.0206,0.0334)"	"(1487,31.0761,29.7186)"	0

References

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Thank You for Your Attention