A million-pound formulation

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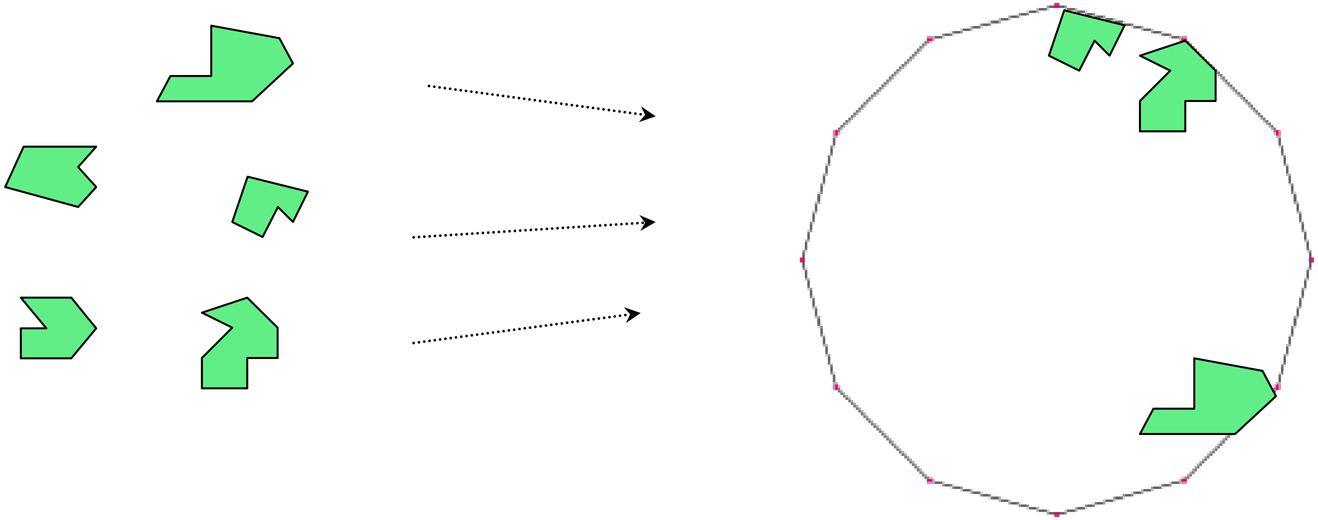
Background..

Eternity (1999), Lord C. Monckton, Ertl Toys.

Task: fill an almost regular dodecagon
(i.e. a polygon with 12 edges)
with 209 irregular polygon pieces.

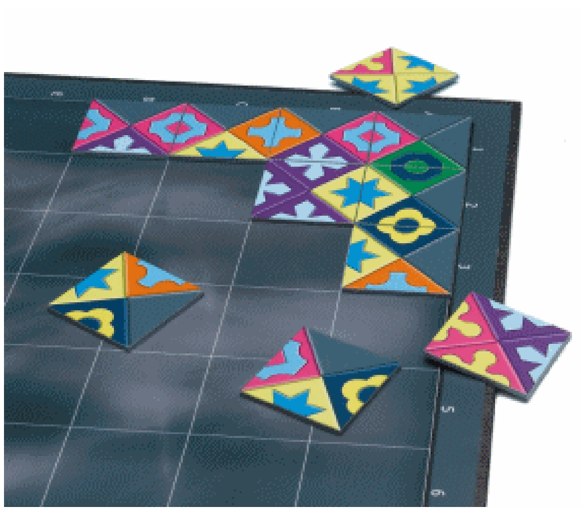


-> Solved 2000 by A. Shelby and O. Riordan (1.000.000£).

Eternity II

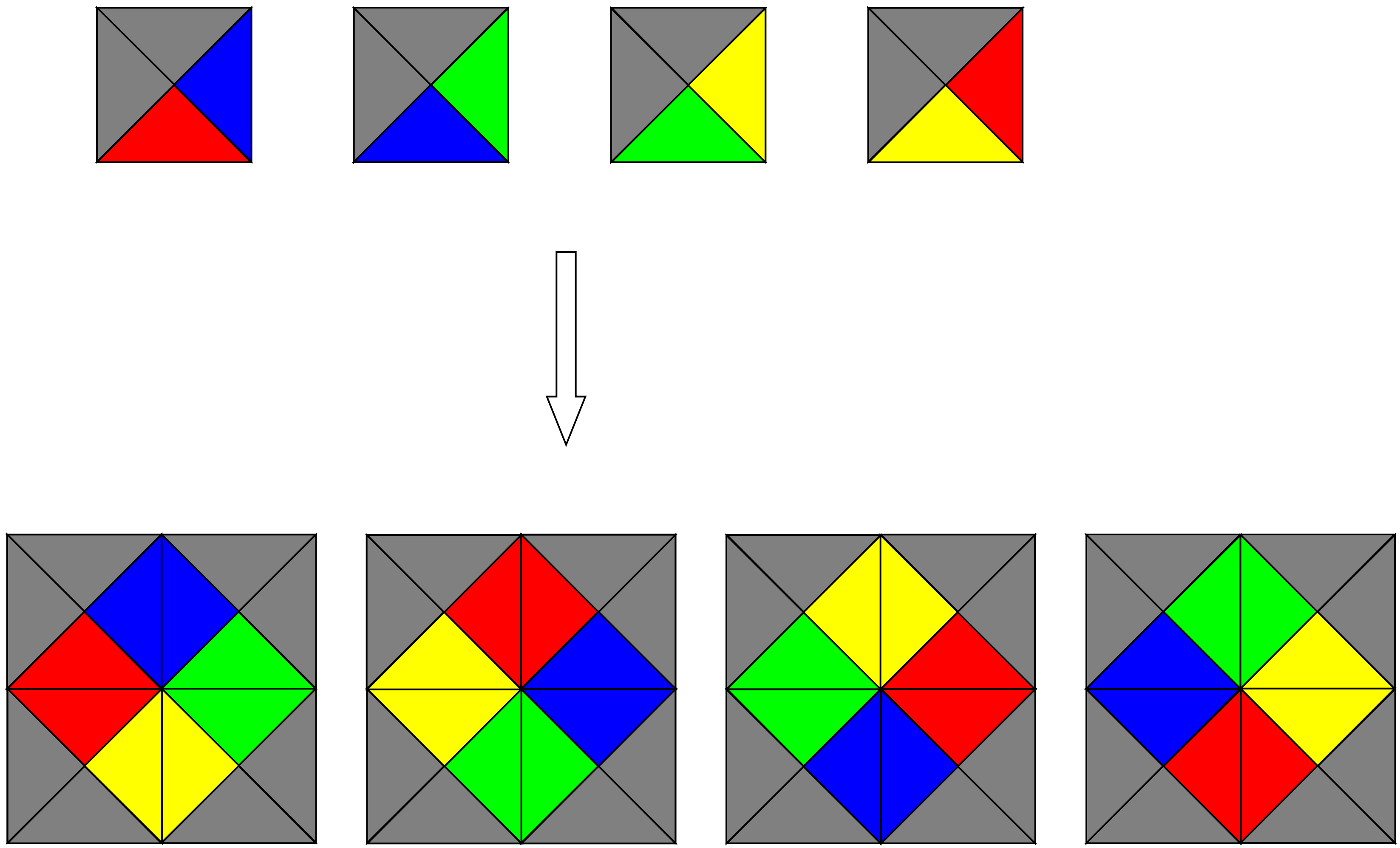
- released 2007 by C. Monckton & Tomy UK Ltd.
- 2.000.000$ prize for the first complete solution (31.12.2008).

Task: fill an 16x16 square with 256 squares, such that all adjacent edges has the same color and pattern. All grey edges should lie at the border of the big square.

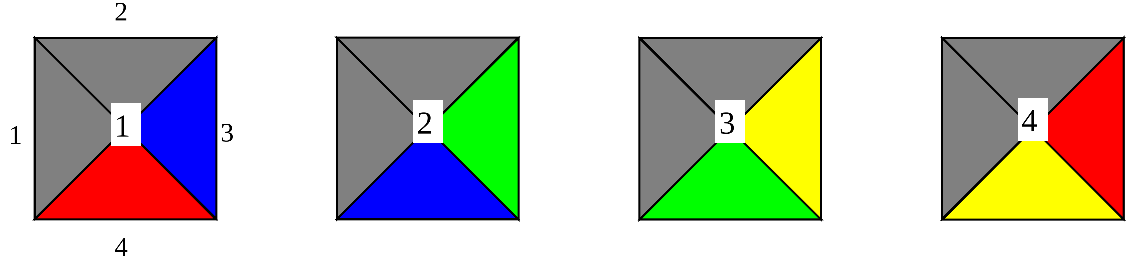


-> no complete solutions obtained, prize went unclaimed
(10.000$ paid to Louis Verhaard, 31.12.2008, 467 matching edges).

The rules.. Set the pieces such that the adjacent edges match and the outer borders are grey. E.g. a 2x2 minipuzzle:



Solution I:



Parameters:

$Face_{kl} \in \{0,1,2,3,4\}$, denotes the pattern on side l of the piece k

(k=1,...,4 and l=1,...,4).

Variables:

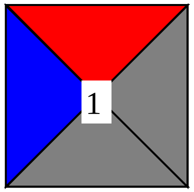
$$Y_{ijk} = \begin{cases} 1 & \text{, if piece } k \text{ is put in position } (i,j) \\ 0 & \text{, otherwise.} \end{cases}$$

i=1,2, j=1,2, k=1,...,4.

$$R_{kl} = \begin{cases} 1 & \text{, if piece } k \text{ is rotated } l * 90^\circ \text{ counterwise} \\ 0 & \text{, otherwise.} \end{cases}$$

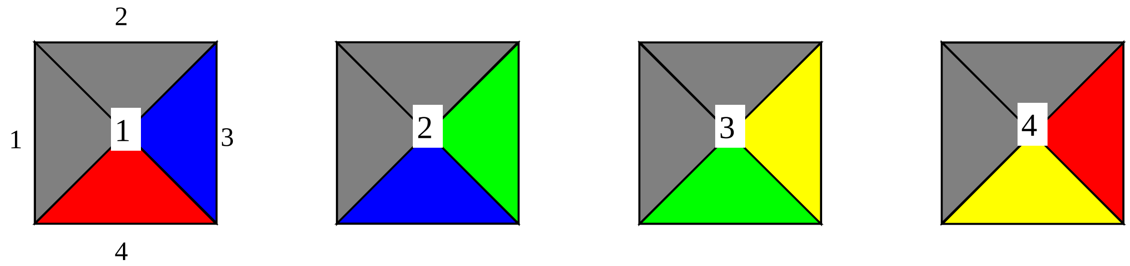
k=1,...,4, l=1,2,3.

$FC_{kl} \in [0,4]$, the pattern of side l of piece k.

Example. $R_{12}=1$: 

Binary variables 2·2·4+4·3 = 28,
Continuous variables 4·4 = 16 + assisting ones

Solution I:



Objective:

$$\max \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{4} Y_{ijk}$$

Subject to:

$$\sum_{k=1}^{4} Y_{ijk} \le 1, \ \forall\ i,\ j,$$

$$\sum_{i=1}^{2}\sum_{j=1}^{2} Y_{ijk} \le 1, \ \forall\ k,$$

$$R_{k1} + R_{k2} + R_{k3} \le 1, \forall k \qquad \text{(rotations)}$$

$$Y_{11k} + Y_{12k} + Y_{21k} + Y_{22k} \le 1, \forall k \qquad \text{(corners)}$$

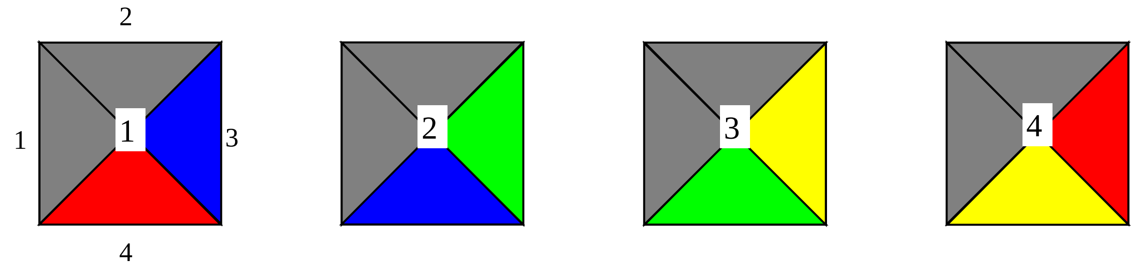
$$FC_{k1} \le M(1 - Y_{11k} - Y_{21k}), \forall k \qquad \text{(borders)}$$

$$FC_{k2} \le M(1 - Y_{11k} - Y_{12k}), \forall k$$

$$FC_{k3} \le M(1 - Y_{12k} - Y_{22k}), \forall k$$

$$FC_{k4} \le M(1 - Y_{21k} - Y_{22k}), \forall k$$

Constraints:



Patterns:

$$FC_{k1} = Face_{k1}(1 - R_{k1} - R_{k2} - R_{k3}) + Face_{k4}R_{k1} + Face_{k3}R_{k2} + Face_{k2}R_{k3},\ \forall k$$

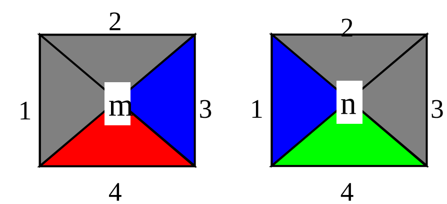
$$FC_{k2} = Face_{k2}(1 - R_{k1} - R_{k2} - R_{k3}) + Face_{k1}R_{k1} + Face_{k4}R_{k2} + Face_{k3}R_{k3},\ \forall k$$

$$FC_{k3} = Face_{k3}(1 - R_{k1} - R_{k2} - R_{k3}) + Face_{k2}R_{k1} + Face_{k1}R_{k2} + Face_{k4}R_{k3},\ \forall k$$

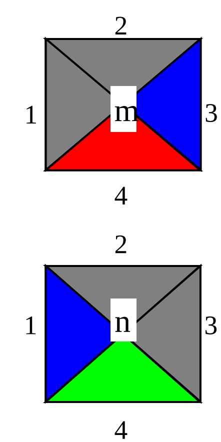
$$FC_{k4} = Face_{k4}(1 - R_{k1} - R_{k2} - R_{k3}) + Face_{k3}R_{k1} + Face_{k2}R_{k2} + Face_{k1}R_{k3},\ \forall k$$

Matching:

$$|FC_{m3} - FC_{n1}| \le M(1 - Y_{11m}Y_{12n} - Y_{21m}Y_{22n}),\ \forall m \ne n$$



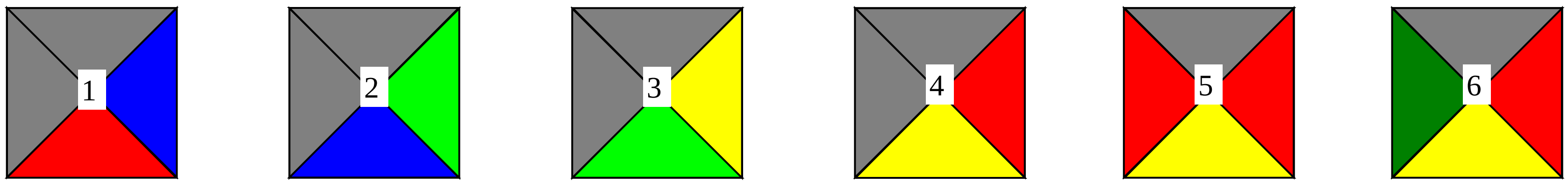
$$|FC_{m4} - FC_{n2}| \le M(1 - Y_{11m}Y_{21n} - Y_{12m}Y_{22n}),\ \forall m \ne n$$



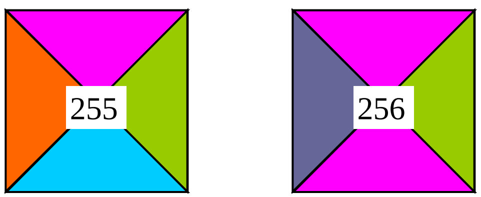
Can be written in a linear form!

→ MILP-problem → global solution

On the complexity of Eternity II, 16×16 puzzle.



. . . etc . . .



Solution I:

$$Y_{ijk} = \begin{cases} 1 & \text{, if piece } k \text{ is put in position } (i,j) \\ 0 & \text{, otherwise.} \end{cases}$$

i=1,...,16, j=1,...,16, k=1,...,256, (= 65.536).

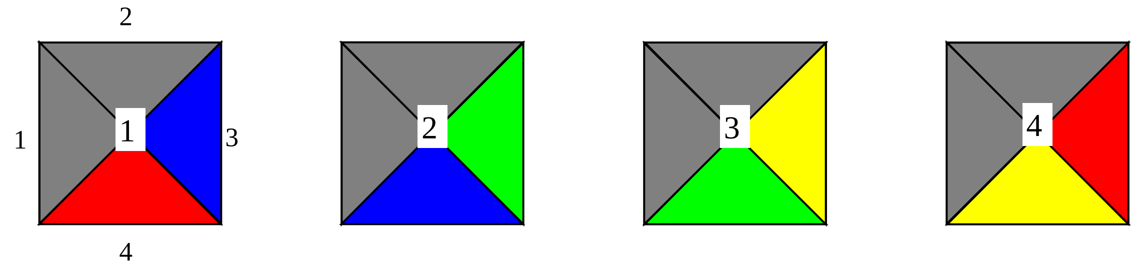
$$R_{kl} = \begin{cases} 1 & \text{, if piece } k \text{ is rotated } l * 90^\circ \text{ counterwise} \\ 0 & \text{, otherwise.} \end{cases}$$

k=1,...,256, l=1,2,3, (=768).

$FC_{kl} \in [0,22]$, the pattern of side l of piece k

i.e. **66.403** binary variables and 1024 continuous + assisting ones.

Solution II:



Variables:

$$X_{ij} = \text{"the number of the piece in position } (i, j)\text{"} \in \{1, \ldots, 256\}$$

$$X_{ij} = 1 + \sum_{k=0}^{7} \beta_{ijk} \cdot 2^k$$

$$\beta_{ijk} \in \{0,1\}, i = 1, \ldots, 16, \; j = 1, \ldots, 16, k = 0,1, \ldots, 7.$$

$$X_{ijl} \in [0,24], \text{ the pattern of side } l \text{ of the piece } k$$

$$R_{kl} = \begin{cases} 1 & \text{, if piece } k \text{ is rotated } l * 90^\circ \text{ counterwise} \\ 0 & \text{, otherwise.} \end{cases} \quad k=1,\ldots,256, \; l=1,2,3.$$

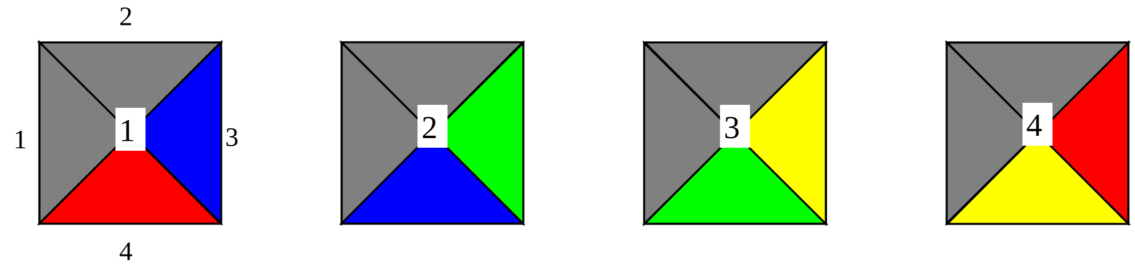
Alternatively; $R_k = \text{"the rotation of piece } k\text{"} = \gamma_{k0} + \gamma_{k1} \cdot 2$

$\gamma_{ki} \in \{0,1\}, i = 0,1, k = 1,2, \ldots, 256.$

→ Binary variables: 16·16·8 + 256·3 = **2816** (cmp 66.403).

Continuous: 16·16·4 = 1024 + assisting ones.

Solution II:



Constraints:

$$\sum_{k=0}^{7} | \beta_{ijk} - \beta_{lmk} | \geq 1,$$

$$i, j, l, m = 1, \ldots, 16, i \neq l, \ j \neq m$$

(all pieces different)

Can be written in a linear form!

Matching:

$$X_{ij3} = X_{i(j+1)1}, i = 1, \ldots, 16, \ j = 1, \ldots, 15.$$

$$X_{ij4} = X_{(i+1)j2}, i = 1, \ldots, 15, \ j = 1, \ldots, 16.$$

Note, that the number of maximal matching is 480.

The patterns:

a) $X_{ijk} = \mathrm{fun}(X_{ij}, R_k), k = 1,2,3,4.$

b) $X_{ij1} = \sum_{k=1}^{256} \left(Face_{k1}(1 - R_{k1} - R_{k2} - R_{k3}) + Face_{k4} R_{k1} + Face_{k3} R_{k2} + Face_{k2} R_{k3} \right) \cdot Pos_{ijk}$

c) $X_{ij1} \geq Face_{k1}(1 - R_{k1} - R_{k2} - R_{k3}) + Face_{k4} R_{k1} + Face_{k3} R_{k2} + Face_{k2} R_{k3} - M(1 - Pos_{ijk})$

$k = 1,2,\ldots,256.$

Variables needed for b) and c): $Pos_{ijk} = \max\left\{ -\left| X_{ij} - k \right| + 1, 0 \right\}$ (Continuous vars)

Can be written in a linear form!

Proposals for future actions (e.g. thesis, projects etc):

- implementation and testing of the different formulations using e.g. GAMS/AlphaECP, CPLEX, GAMS/LindoGlobal etc.
- optimization of the formulation and coding
- testing and comparisons with heuristics
- study of literature of similar problems (maximal matching, maximal satisfaction etc).