On comparison of different approaches to the stability radius calculation

Olga Karelkina

University of Turku 2011

Outline

- Preliminaries
- Problem statement
- Exact method for calculation stability radius proposed by Chakravarti and Wagelmans
- NSGA-II adaptation for calculation stability radius
- Illustration and comparison of two approaches

Two major directions of investigation can be single out

- quantitative
  - bounds for feasible changes in initial data, which preserve some pre-assigned properties of optimal solutions
  - deriving algorithms for the bounds calculation
- qualitative
  - conditions under which the set of optimal solutions of the problem possesses a certain pre-assigned property of invariance to external influence on initial data of the problem

Shortest path problem (SP)

Given a directed graph $G=(V,E), |V|=m$ and $|E|=n$

c_i – a nonnegative cost associated with each edge $e_i \in E$

Problem: find a directed path from a source node s to a distinguished terminal node t, with the minimum total cost.

The feasible set is the set of all sequences $P=(e_{i_1},\ldots,e_{i_k})$, these sequences are directed paths from s to t in G.

Cost mapping

$$c(P)=\sum_{i=1}^{k} c_i$$

SP as LP

Vector of ordered edges costs

$$C = (c_1, c_2, \ldots, c_n) \in \mathbf{R}_+^n, \quad x = (x_1, x_2, \ldots, x_n) \in \mathbf{E}^n$$

$$x_i = \begin{cases} 1, & if \quad e_i \in P, \\ 0 & otherwise \end{cases}$$

$$\sum_{e_i \in E} c_i x_i \to \min$$

$$\sum_{e: e_i \to j} x_i - \sum_{e: e_i \leftarrow j} x_i = \begin{cases} 1, & if \quad j = s, \\ -1, & if \quad j = t, \\ 0 & otherwise \end{cases}$$

Perturbation of the problem

We define norms l_1 and l_∞ in $\mathbf{R}^d$ for any finite dimension $d \in \mathbf{N}$

$$\|y\|_1 = \sum_{i \in N_d} |y_i|, \qquad \|y\|_\infty = \max\left\{|y_i| \mid i \in N_d\right\},$$

$$y = (y_1, y_2, \ldots, y_d)^T \in \mathbf{R}^d,\ N_d = 1, 2, \ldots, d.$$

The perturbation of the problem parameters is modeled by adding to the cost vector C perturbing vector

$$C' = \left(c'_1, c'_2, \ldots, c'_n\right) \in \mathbf{R}^n,\ \|C'\|_\infty < \varepsilon,\ \varepsilon > 0.$$

The set of the perturbing vectors is denoted by $\Omega(\varepsilon)$.

Stability radius

Let $X \subset 2^{E^n}$ be the set of feasible solutions to the shortest path problem

Let $X_{opt}(C)$ be the set of optimal solutions to the shortest path problem with cost vector C .

An optimal solution $x \in X_{opt}(C)$ is called stable if

$$\exists \varepsilon > 0 \quad \forall C' \in \Omega(\varepsilon) \quad x \in X_{opt}(C + C').$$

Stability radius of an optimal solution $x \in X_{opt}(C)$

$$\rho(x, C) = \begin{cases} \sup \Theta, & if \quad \Theta \neq \varnothing, \\ 0, & if \quad \Theta = \varnothing. \end{cases}$$

$$\Theta = \left\{ \varepsilon > 0 \quad | \quad \forall C' \in \Omega(\varepsilon) \quad \left(x \in X_{opt}(C + C') \right) \right\}.$$

Stability radius

V.A. Emelichev, V.N. Krichko, D.P. Podkopaev, On the radius of stability of a vector problem of linear Boolean programming, Discrete Math. Appl. 10 (2000) 103 – 108

$$\rho(x, C) = \min_{x' \in X \setminus \{x\}} \frac{\sum_{i \in N_n} c_i (x'_i - x_i)}{\left\| x' - x \right\|_1} \qquad \textbf{(1)}$$

The largest ρ such that for $\left|c'_i\right| \le \rho, i \in N_n$

$$\sum_{i \in N_n} \left(c_i + c'_i\right) x_i \le \sum_{i \in N_n} \left(c_i + c'_i\right) x'_i, \quad \forall x' \in X$$

Calculating the stability radii of an optimal solution to the linear problem of 0-1 programming

$$Cx \to \min_{x \in X} \qquad \textbf{(2)}$$

Theorem Let x be an optimal solution to (2). The stability radius of x is the maximum number ρ satisfying the following inequality :

$$\min_{x' \in X \setminus \{x\}} \left\{ \sum_{i \in N_n} (c_i - \rho d_i) x'_i \right\} \geq \sum_{i \in N_n} (-c_i + \rho) x_i \qquad \textbf{(3)}$$

$$d_i = \begin{cases} 1, & if \quad x_i = 0, \\ -1, & if \quad x_i = 1. \end{cases}$$

$\rho(x, C)$ is the maximal ρ satisfying the inequality :

$$\rho \leq \min_{x' \in X \setminus \{x\}} \frac{\sum_{i \in N_n} C_i (x'_i - x_i)}{\|x' - x\|_1}$$

From here taking into account

$$|x'_i - x_i| = x + d_i x'_i, \quad \forall i \in N_n$$

$$\|x' - x\|_1 = \sum_{i \in N_n} |x'_i - x_i| = \sum_{i \in N_n} \left(x + d_i x'_i \right)$$

we get

$$\min_{x' \in X \setminus \{x\}} \left\{ \sum_{i \in N_n} \left(c_i - \rho d_i \right) x'_i \right\} \geq \sum_{i \in N_n} \left(-c_i + \rho \right) x_i$$

Let us denote

$$v(\rho) = \min_{x' \in X \setminus \{x\}} \left\{ \sum_{i \in N_n} (c_i - \rho d_i) x'_i \right\}$$

D. Gusfield, Parametric combinatorial computing and a problem of program module distribution, J. Assoc. Comput. Mach. 30 (1983) 551 – 563

$v(\rho)$ is a continuous, piecewise linear and concave function of ρ

Lemma The number of linear pieces of $v(\rho)$ is $\mathrm{O}(n^2)$

Chakravarti and Wagelmans polinomial algorithm

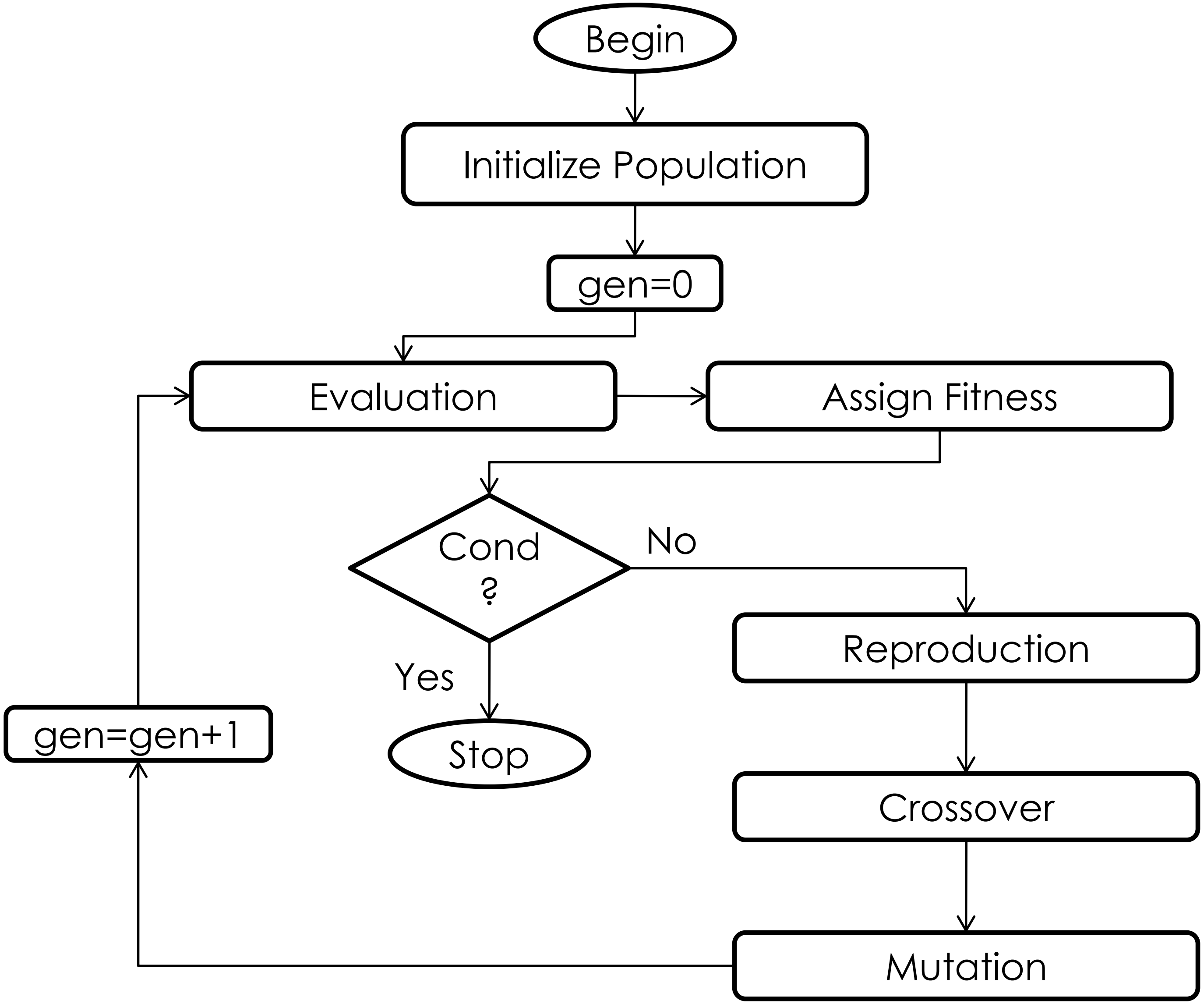
Construction of $v(\rho)$ on $[0, \|C\|_\infty]$

- Compute $v(0)$ and $v(\|C\|_\infty)$
- The optimal solutions associated with these values each defines a linear function on $[0, \|C\|_\infty]$
- If these functions are identical, then $v(\rho)$ is simply this linear function
- Otherwise, we have two linear functions which intersect at a unique value $\bar{\rho} \in [0, \|C\|_\infty]$
- If $(\bar{\rho}, v(\bar{\rho}))$ coincides with the intersection point, then $v(\rho)$ is the concave lower envelope of the two linear functions
- Otherwise, the optimal solution associated with $\bar{\rho}$ defines a third linear function which intersects each of the other linear functions on $[0, \|C\|_\infty]$

A fast and elitist multi-objective genetic algorithm: NSGA-II

Modules

A. A fast non-dominated sorting approach
B. Diversity presentation
    - Density estimation
    - Crowded comparison operator
C. The main loop


Begin
Initialize Population
gen=0
Evaluation
Assign Fitness
Cond ?
No
Yes
Stop
Reproduction
Crossover
Mutation
gen=gen+1

Implementation of NSGA-II into calculation stability radius

$$\sum_{i\in N_n} c_i(x_i' - x_i) = f_1(x,C) \to \min$$

$$\|x' - x\|_1 = f_2(x,C) \to \max$$

Pareto set

$$P^2(C) = \{x \in X \mid \forall x' \in X$$

$$\Big(\big(f_1(x,C) \le f_1(x',C) \wedge f_2(x,C) \ge f_2(x',C)\big) \wedge$$

$$\wedge\big(f_1(x,C) \ne f_1(x',C) \vee f_2(x,C) \ne f_2(x',C)\big)\Big)$$

Representation

- Graph is represented by costs matrix (vector)
- Every variable (feasible solution) is coded in a fixed length binary string

Initialization

- Breadth First Search

Evaluation

- A fast non-dominated sorting approach

$P' =$ find-nondominated-front(P)	
$P' = \{1\}$	include first member in P'
for each $p \in P \wedge p \notin P'$	take one soltion at a time
$P' = P' \mathrm{U} \{p\}$	include p in P' temporarily
for each $q \in P' \wedge q \neq p$	compare p with ppther members of P'
if $p \text{ p } q$, then $P' = P' \setminus \{q\}$	if p dominates a member of P', delete it
else if $p \text{ p } q$, then $P' = P' \setminus \{p\}$	if p is dominated by other members of P', do not include P' in

Assign fitness

- Density estimation

  Crowding distance $i_{distance}$ is an estimate of the size of the largest cuboid enclosing the point i without including any pther point in the population

- Crowded comparison operator

$$i \ \mathrm{p}_{n} \ j \Leftrightarrow \left(i_{rank} < j_{rank}\right) \vee \left(\left(i_{rank} = j_{rank}\right) \wedge \left(i_{distance} > j_{distance}\right)\right)$$

Reproduction

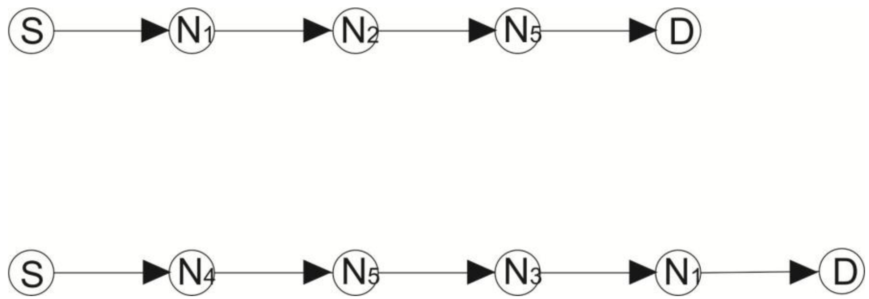
- The tournament selection scheme

The strings with minimum front number and minimum ratios

$$\frac{f_1(x,C)}{f_2(x,C)}$$

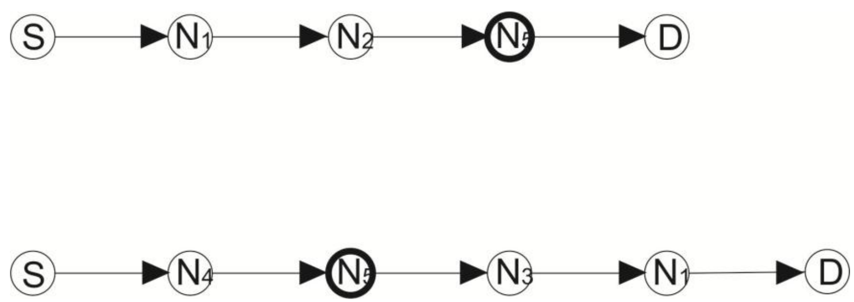
are selected to the mating pool.

Crossovers

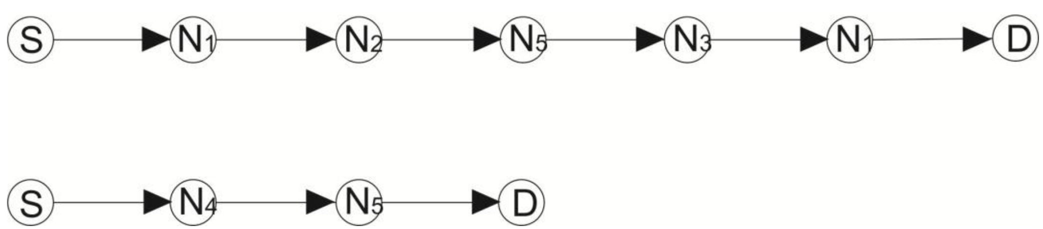


- One-Node crossover

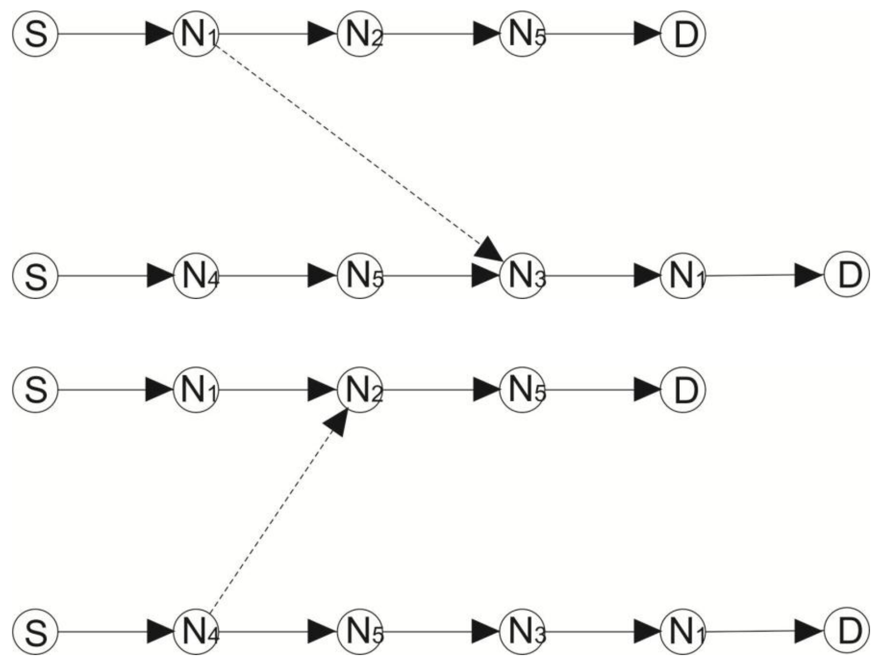
Chromosomes before crossover



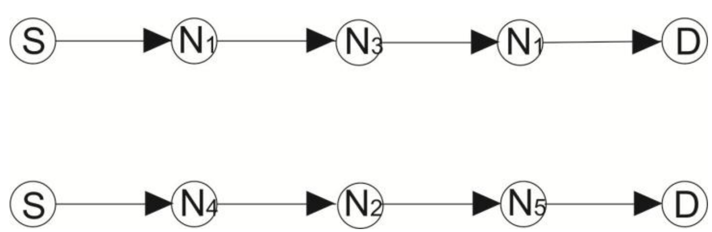
Chromosomes after crossover



- One-Edge crossover
  Chromosomes before crossover

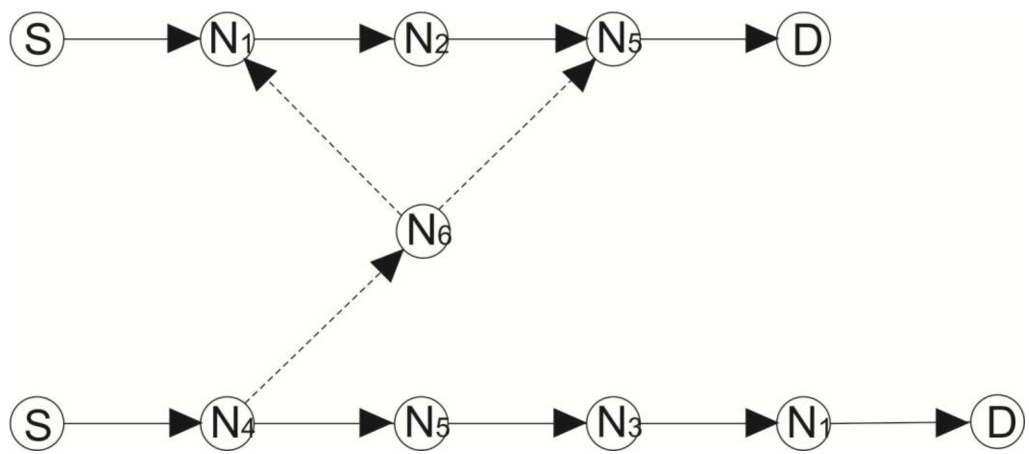


Chromosomes after crossover

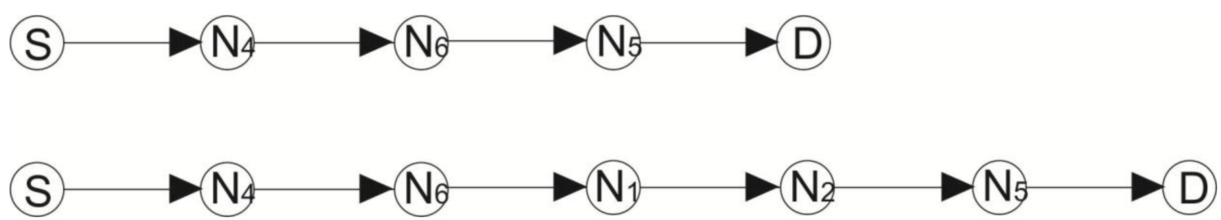


- One-Node-Two-Edges crossover

Chromosomes before crossover

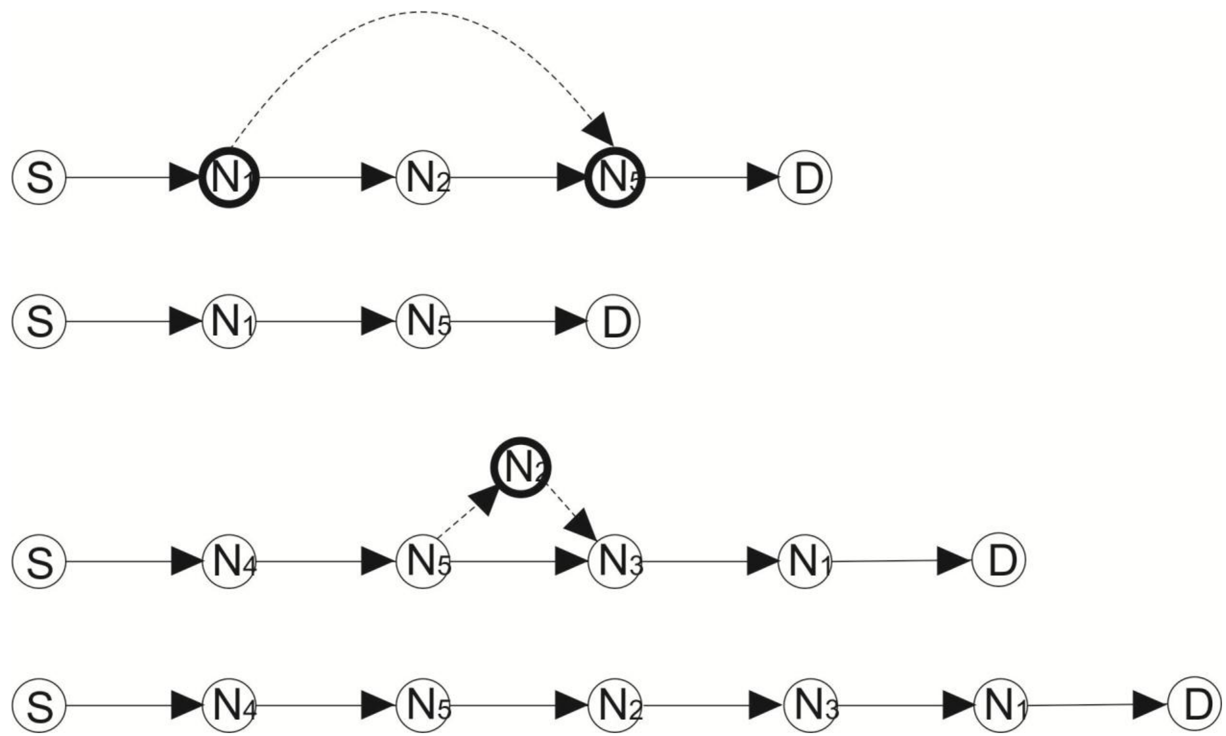


Chromosomes after crossover

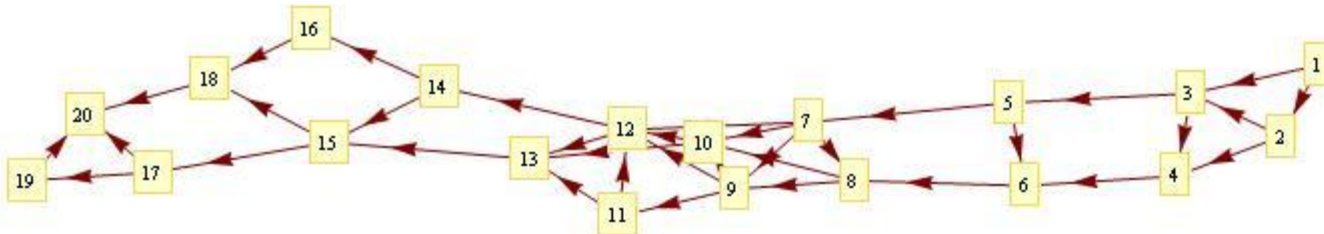


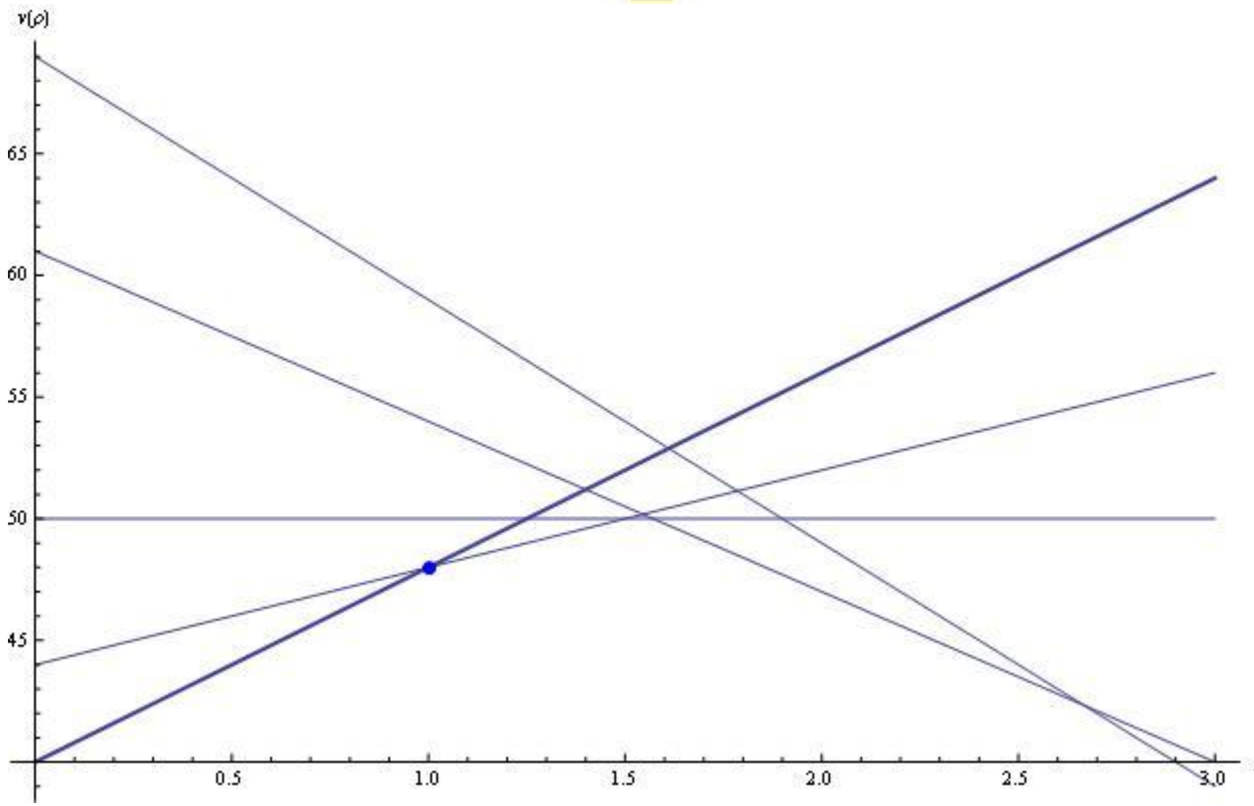
Mutation

- Two mutation types



Simulation results





References

1. V.A. Emelichev, V.N. Krichko, D.P. Podkopaev, On the radius of stability of a vector problem of linear Boolean programming, Discrete Math. Appl. 10 (2000) 103 – 108
2. N. Chakravarti, A. P.M. Wagelmans, Calculation of stability radii for combinatorial optimization problems, OR Letters. 23 (1998) 1 – 7
3. D. Gusfield, Parametric combinatorial computing and a problem of program module distribution, J. Assoc. Comput. Mach. 30 (1983) 551 – 563
4. V. A. Emelichev, D.P. Podkopaev, Quantitative stability analysis for vector problems of 0 – 1 programming, Discrete Optimization. 7 (2010) 48 – 63
5. K. Deb, A. Pratap, S. Agarwal, T. Meyarivan, A fast and elitist multi-objective genetic algorithm: NSGA-II, Evolutionary Computation. 6 (2) (2002), 182 – 197

Thank You for Your interest