On comparison of different approaches to the stability radius calculation

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Outline

- Preliminaries
- Problem statement
- Exact method for calculation stability radius proposed by Chakravarti and Wagelmans
- NSGA-II adaptation for calculation stability radius
- Illustration and comparison of two approaches
Two major directions of investigation can be single out

- quantitative
  - bounds for feasible changes in initial data, which preserve some pre-assigned properties of optimal solutions
  - deriving algorithms for the bounds calculation

- qualitative
  - conditions under which the set of optimal solutions of the problem possesses a certain pre-assigned property of invariance to external influence on initial data of the problem
Shortest path problem (SP)

Given a directed graph \( G = (V, E) \), \(|V| = m\) and \(|E| = n\)

\(c_i\) – a nonnegative cost associated with each edge 
\(e_i \in E\)

Problem: find a directed path from a source node \(s\) 
to a distinguished terminal node \(t\), with the minimum 
total cost.

The feasible set is the set of all sequences 
\(P = (e_{i_1}, K, e_{i_k})\), 
these sequences are directed paths from \(s\) to \(t\) in \(G\).

Cost mapping

\[
c(P) = \sum_{i=1}^{k} c_i
\]
SP as LP

Vector of ordered edges costs

\[ C = (c_1, c_2, K, c_n) \in \mathbb{R}_+^n, \quad x = (x_1, x_2, K, x_n) \in \mathbb{E}^n \]

\[ x_i = \begin{cases} 1, & \text{if } e_i \in P, \\ 0 & \text{otherwise} \end{cases} \]

\[ \sum_{e_i \in E} c_i x_i \rightarrow \text{min} \]

\[ \sum_{e:e_i \rightarrow j} x_i - \sum_{e:e_i \leftarrow j} x_i = \begin{cases} 1, & \text{if } j = s, \\ -1, & \text{if } j = t, \\ 0 & \text{otherwise} \end{cases} \]
Perturbation of the problem

We define norms $l_1$ and $l_\infty$ in $\mathbb{R}^d$ for any finite dimension $d \in \mathbb{N}$

$$
\|y\|_1 = \sum_{i \in N_d} |y_i|, \quad \|y\|_\infty = \max \{|y_i| : i \in N_d\},
$$

$$y = (y_1, y_2, K, y_d)^T \in \mathbb{R}^d, \; N_d = 1, 2, K, d.
$$

The perturbation of the problem parameters is modeled by adding to the cost vector $C$ perturbing vector $C' = (c'_1, c'_2, K, c'_n) \in \mathbb{R}^n, \; \|C'\|_\infty < \varepsilon, \; \varepsilon > 0.$

The set of the perturbing vectors is denoted by $\Omega(\varepsilon)$. 

Stability radius

Let \( X \subset 2^E \) be the set of feasible solutions to the shortest path problem.

Let \( X_{opt}(C) \) be the set of optimal solutions to the shortest path problem with cost vector \( C \).

An optimal solution \( x \in X_{opt}(C) \) is called stable if

\[
\exists \varepsilon > 0 \quad \forall C' \in \Omega(\varepsilon) \quad x \in X_{opt}(C + C').
\]

Stability radius of an optimal solution \( x \in X_{opt}(C) \)

\[
\rho(x, C) = \begin{cases} 
\sup \Theta, & \text{if } \Theta \neq \emptyset, \\
0, & \text{if } \Theta = \emptyset.
\end{cases}
\]

\( \Theta = \left\{ \varepsilon > 0 \mid \forall C' \in \Omega(\varepsilon) \quad \left( x \in X_{opt}(C + C') \right) \right\}. \)
Stability radius


\[ \rho(x, C) = \min_{x' \in X \setminus \{x\}} \frac{\sum c_i(x'_i - x_i)}{\|x' - x\|_1} \]  

(1)

The largest \( \rho \) such that for \( |c'_i| \leq \rho, i \in N_n \)

\[ \sum_{i \in N_n} (c_i + c'_i) x_i \leq \sum_{i \in N_n} (c_i + c'_i) x'_i, \quad \forall x' \in X \]
Calculating the stability radii of an optimal solution to the linear problem of 0-1 programming

$$C x \rightarrow \min_{x \in X}$$ \hspace{1cm} (2)

**Theorem** Let $x$ be an optimal solution to (2). The stability radius of $x$ is the maximum number $\rho$ satisfying the following inequality:

$$\min_{x' \in X \setminus \{x\}} \left\{ \sum_{i \in N_n} (c_i - \rho d_i) x'_i \right\} \geq \sum_{i \in N_n} (-c_i + \rho) x_i \hspace{1cm} (3)$$

$$d_i = \begin{cases} 1, & \text{if } x_i = 0, \\ -1, & \text{if } x_i = 1. \end{cases}$$
\( \rho(x, C) \) is the maximal \( \rho \) satisfying the inequality:

\[
\rho \leq \min_{x' \in X \setminus \{x\}} \frac{\sum_{i \in N_n} C_i (x'_i - x_i)}{\|x' - x\|_1}
\]

From here taking into account

\[
|x'_i - x_i| = x + d_i x'_i, \quad \forall i \in N_n
\]

\[
\|x' - x\|_1 = \sum_{i \in N_n} |x'_i - x_i| = \sum_{i \in N_n} (x + d_i x'_i)
\]

we get

\[
\min_{x' \in X \setminus \{x\}} \left\{ \sum_{i \in N_n} \left( c_i - \rho d_i \right) x'_i \right\} \geq \sum_{i \in N_n} (-c_i + \rho) x_i
\]
Let us denote

\[ v(\rho) = \min_{x' \in X \setminus \{x\}} \left\{ \sum_{i \in N_n} (c_i - \rho d_i) x'_i \right\} \]


\[ v(\rho) \] is a continuous, piecewise linear and concave function of \( \rho \)

**Lemma** The number of linear pieces of \( v(\rho) \) is \( O(n^2) \)
Chakravarti and Wagelmans polynomial algorithm

Construction of \( v(\rho) \) on \([0, \|C\|_\infty]\)

- Compute \( v(0) \) and \( v(\|C\|_\infty) \)
- The optimal solutions associated with these values each defines a linear function on \([0, \|C\|_\infty]\)
- If these functions are identical, then \( v(\rho) \) is simply this linear function
- Otherwise, we have two linear functions which intersect at a unique value \( \bar{\rho} \in [0, \|C\|_\infty] \)
- If \( (\bar{\rho}, v(\bar{\rho})) \) coincides with the intersection point, then \( v(\rho) \) is the concave lower envelope of the two linear functions
- Otherwise, the optimal solution associated with \( \bar{\rho} \) defines a third linear function which intersects each of the other linear functions on \([0, \|C\|_\infty]\)
A fast and elitist multi-objective genetic algorithm: NSGA-II

Modules

A. A fast non-dominated sorting approach
B. Diversity presentation
   • Density estimation
   • Crowded comparison operator
C. The main loop
Begin

Initialize Population

gen = 0

Evaluation

Assign Fitness

Cond?

Reproduction

Mutation

Crossover

Stop

Yes

No

gen = gen + 1

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Implementation of NSGA-II into calculation stability radius

\[ \sum_{i \in N_n} c_i(x'_i - x_i) = f_1(x, C) \rightarrow \min \]

\[ \|x' - x\|_1 = f_2(x, C) \rightarrow \max \]

Pareto set

\[ P^2(C) = \{ x \in X | \forall x' \in X \] \]

\[ ((f_1(x, C) \leq f_1(x', C) \land f_2(x, C) \geq f_2(x', C)) \land\]

\[ \land (f_1(x, C) \neq f_1(x', C) \lor f_2(x, C) \neq f_2(x', C))) \]
Representation

- Graph is represented by costs matrix (vector)
- Every variable (feasible solution) is coded in a fixed length binary string

Initialization

- Breadth First Search

Evaluation

- A fast non-dominated sorting approach

\[ P' = \text{find-nondominated-front}(P) \]
\[ P' = \{1\} \]
for each \( p \in P \land p \notin P' \)
\[ P' = P' \cup \{ p \} \]
for each \( q \in P' \land q \neq p \)
if \( p \) dominates \( q \), then \( P' = P' \setminus \{ q \} \)
else if \( p \) dominates another member of \( P' \), delete it

(include first member in \( P' \))
(take one solution at a time)
(include \( p \) in \( P' \) temporarily)
(compare \( p \) with other members of \( P' \))
(if \( p \) dominates another member of \( P' \), delete it)

do not include \( P' \) in
Assign fitness

- Density estimation

  Crowding distance $d_{distance}$ is an estimate of the size of the largest cuboid enclosing the point $i$ without including any other point in the population.

- Crowded comparison operator

  \[
  i \preceq_p j \iff (r_{rank_i} < r_{rank_j}) \lor ((r_{rank_i} = r_{rank_j}) \land (d_{distance_i} > d_{distance_j}))
  \]
Reproduction

- The tournament selection scheme

  The strings with minimum front number and minimum ratios

\[
\frac{f_1(x, C)}{f_2(x, C)}
\]

are selected to the mating pool.
Crossovers

- One-Node crossover

Chromosomes before crossover

Chromosomes after crossover
- One-Edge crossover

Chromosomes before crossover

Chromosomes after crossover
- One-Node-Two-Edges crossover

Chromosomes before crossover

\[ S \rightarrow N_1 \rightarrow N_2 \rightarrow N_5 \rightarrow D \]

\[ S \rightarrow N_4 \rightarrow N_3 \rightarrow N_1 \rightarrow D \]

Chromosomes after crossover

\[ S \rightarrow N_4 \rightarrow N_6 \rightarrow N_5 \rightarrow D \]

\[ S \rightarrow N_4 \rightarrow N_6 \rightarrow N_1 \rightarrow N_2 \rightarrow N_5 \rightarrow D \]
Mutation

- Two mutation types
Simulation results
References

Thank You for Your interest