Extremality, Stationarity and Regularity of Collections of Sets

Alexander Kruger

Centre for Informatics and Applied Optimization,
School of Science, Information Technology & Engineering
University of Ballarat, Australia
a.kruger@ballarat.edu.au

Coauthor: Marco López Cerda

Turku, 28 June 2012

Collections of Sets

- Convex case
  - Separation theorem

Collections of Sets

- Convex case
  - Separation theorem
  - (Bounded) linear regularity (Bauschke, Borwein, 1993; Ng, Yang, 2004; Burke, Deng, 2005)

Collections of Sets

- Convex case
  - Separation theorem
  - (Bounded) linear regularity (Bauschke, Borwein, 1993; Ng, Yang, 2004; Burke, Deng, 2005)
- Nonconvex case
  - Dubovitskii–Milyutin formalism (1965)

Collections of Sets

- Convex case
  - Separation theorem
  - (Bounded) linear regularity (Bauschke, Borwein, 1993; Ng, Yang, 2004; Burke, Deng, 2005)
- Nonconvex case
  - Dubovitskii–Milyutin formalism (1965)
  - Extremal principle (Kruger, Mordukhovich, 1980)

Collections of Sets

- Convex case
  - Separation theorem
  - (Bounded) linear regularity (Bauschke, Borwein, 1993; Ng, Yang, 2004; Burke, Deng, 2005)
- Nonconvex case
  - Dubovitskii–Milyutin formalism (1965)
  - Extremal principle (Kruger, Mordukhovich, 1980)
  - Boundary condition, nonconvex separation property (Borwein, Jofré, 1998)
  - Jamesons property (G) (1972)
  - Metric inequality (Ioffe, 1989; Ngai, Théra, 2001)
  - (Strong) conical hull intersection property (Chui, Deutsch, Ward, 1992; Deutsch, Li, Ward; 1997)

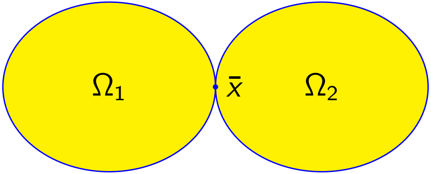
Outline

1. Finite Collections
   - Extremal Collection of Sets
   - Extremal Principle
   - Stationarity vs Regularity

2. Infinite Collections
   - Stationarity vs Regularity
   - Intersection Rule
   - Constrained Optimization

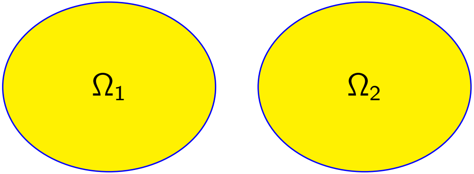
Stationarity and Regularity of Finite Collections

Extremal Collection of Sets (Kruger, Mordukhovich, 1980)



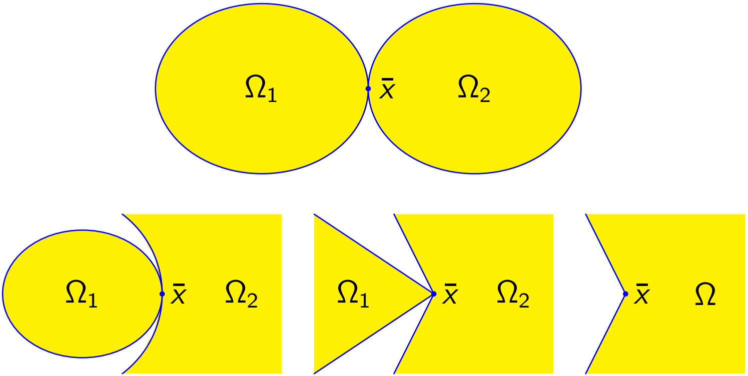
Stationarity and Regularity of Finite Collections

Extremal Collection of Sets (Kruger, Mordukhovich, 1980)



Stationarity and Regularity of Finite Collections

Extremal Collection of Sets (Kruger, Mordukhovich, 1980)



Stationarity and Regularity of Finite Collections

Extremal Collection of Sets

X – Banach space,

$\mathbf{\Omega} := \{\Omega_i\}_{i \in I} \subset X, \quad 1 < |I| < \infty, \quad \bar{x} \in \bigcap_{i \in I} \Omega_i$

X – Banach space,
$\mathbf{\Omega} := \{\Omega_i\}_{i\in I} \subset X, \quad 1 < |I| < \infty, \quad \bar{x} \in \bigcap_{i\in I} \Omega_i$

Definition

$\mathbf{\Omega}$ is *locally extremal* at $\bar{x}$ if $\exists \rho > 0\ \forall \varepsilon > 0\ \exists a_i \in X$, $i \in I$,

$$\max_{i\in I} \|a_i\| < \varepsilon \quad \text{and} \quad \bigcap_{i\in I} (\Omega_i - a_i) \bigcap B_\rho(\bar{x}) = \emptyset$$

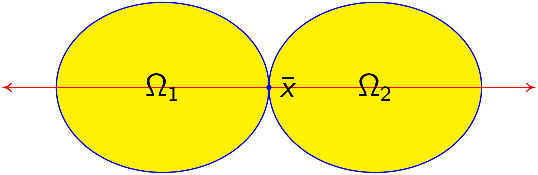
X – Banach space,
$\mathbf{\Omega} := \{\Omega_i\}_{i\in I} \subset X, \quad 1 < |I| < \infty, \quad \bar{x} \in \bigcap_{i\in I} \Omega_i$

Definition

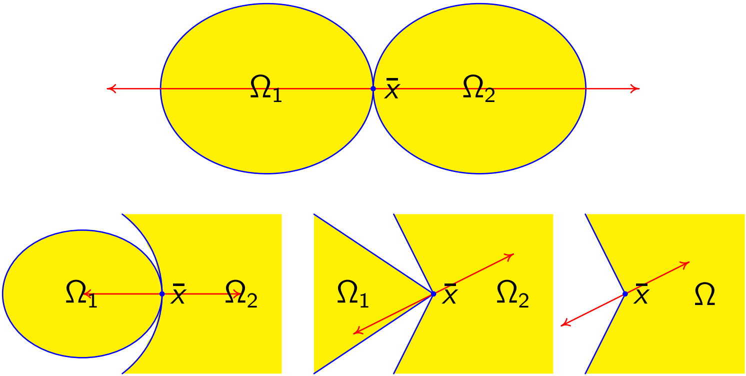
$\mathbf{\Omega}$ is *locally extremal* at $\bar{x}$ if $\exists \rho > 0\ \forall \varepsilon > 0\ \exists a_i \in X,\ i \in I,$

$$\max_{i\in I} \|a_i\| < \varepsilon \quad \text{and} \quad \bigcap_{i\in I} (\Omega_i - a_i) \bigcap B_\rho(\bar{x}) = \emptyset$$

$$\theta_\rho[\mathbf{\Omega}](\bar{x}) := \sup\Big\{ r \geq 0 : \bigcap_{i\in I} (\Omega_i - a_i) \bigcap B_\rho(\bar{x}) \neq \emptyset, \forall a_i \in r\mathbb{B} \Big\} = 0$$



Ω_1
$\bar{x}$
Ω_2


Ω_1
$\bar{x}$
Ω_2
Ω_1
$\bar{x}$
Ω_2
Ω_1
$\bar{x}$
Ω_2
$\bar{x}$
Ω

Stationarity and Regularity of Finite Collections

Fréchet Normal Cone

$\bar{x} \in \Omega$

Fréchet normal cone:

$$N_\Omega(\bar{x}) = \left\{ x^* \in X^* : \ \limsup_{x \overset{\Omega}{\to} \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \le 0 \right\}$$

Stationarity and Regularity of Finite Collections

Extremal Principle [Kruger, Mordukhovich (1980); Mordukhovich, Shao (1996)]

Extremal Principle

Ω_i, $i \in I$, are closed. If $\mathbf{\Omega}$ is locally extremal at $\bar{x}$ then $\forall \varepsilon > 0$ $\exists x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$, $x_i^* \in N_{\Omega_i}(x_i)$ $(i \in I)$

$$\left\| \sum_{i \in I} x_i^* \right\| < \varepsilon \sum_{i \in I} \|x_i^*\|$$

Extremal Principle

Ω_i, $i \in I$, are closed. If $\boldsymbol{\Omega}$ is locally extremal at $\bar{x}$ then $\forall \varepsilon > 0$
$\exists x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$, $x_i^* \in N_{\Omega_i}(x_i)$ $(i \in I)$

$$\left\| \sum_{i \in I} x_i^* \right\| < \varepsilon \sum_{i \in I} \|x_i^*\|$$

Theorem

Extremal Principle holds if and only if X is Asplund

Stationarity and Regularity of Finite Collections

Extremal Principle [Kruger, Mordukhovich (1980); Mordukhovich, Shao (1996)]

Extremal Principle

Ω_i, $i \in I$, are closed. If $\mathbf{\Omega}$ is locally extremal at $\bar{x}$ then $\forall \varepsilon > 0$ $\exists x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$, $x_i^* \in N_{\Omega_i}(x_i)$ $(i \in I)$

$$\left\| \sum_{i \in I} x_i^* \right\| < \varepsilon \sum_{i \in I} \|x_i^*\|$$

Theorem

Extremal Principle holds if and only if X is Asplund

$$\hat{\eta}[\mathbf{\Omega}](\bar{x}) := \liminf_{\substack{x_i \xrightarrow{\Omega_i} \bar{x},\, x_i^* \in N_{\Omega_i}(x_i)\ (i \in I) \\ \sum_{i \in I} \|x_i^*\| = 1}} \left\| \sum_{i \in I} x_i^* \right\| = 0$$

Definition

$\mathbf{\Omega}$ is *approximately stationary* at $\bar{x}$ if $\forall\varepsilon > 0\ \exists\rho \in (0, \varepsilon)$, $\omega_i \in \Omega_i \cap B_\varepsilon(\bar{x})$, $a_i \in X\ (i \in I)$

$$\max_{i\in I} \|a_i\| < \varepsilon\rho \quad \text{and} \quad \bigcap_{i\in I}(\Omega_i - \omega_i - a_i)\bigcap(\rho\mathbb{B}) = \emptyset$$

Definition

$\mathbf{\Omega}$ is *approximately stationary* at $\bar{x}$ if $\forall\varepsilon > 0\ \exists\rho \in (0, \varepsilon)$, $\omega_i \in \Omega_i \cap B_\varepsilon(\bar{x})$, $a_i \in X$ $(i \in I)$

$$\max_{i\in I} \|a_i\| < \varepsilon\rho \quad \text{and} \quad \bigcap_{i\in I}(\Omega_i - \omega_i - a_i) \bigcap (\rho\mathbb{B}) = \emptyset$$

Local extremality $\Rightarrow$ approximate stationarity

Definition

$\mathbf{\Omega}$ is *approximately stationary* at $\bar{x}$ if $\forall\varepsilon > 0\ \exists\rho \in (0,\varepsilon)$, $\omega_i \in \Omega_i \cap B_\varepsilon(\bar{x})$, $a_i \in X\ (i \in I)$

$$\max_{i\in I} \|a_i\| < \varepsilon\rho \quad \text{and} \quad \bigcap_{i\in I}(\Omega_i - \omega_i - a_i) \bigcap (\rho\mathbb{B}) = \emptyset$$

Local extremality $\Rightarrow$ approximate stationarity

$$\hat{\theta}[\mathbf{\Omega}](\bar{x}) := \liminf_{\substack{\omega_i \xrightarrow{\Omega_i} \bar{x} \\ \rho\to+0}} \frac{\theta_\rho[\{\Omega_i - \omega_i\}_{i\in I}](0)}{\rho} = 0$$

Definition

$\mathbf{\Omega}$ is

- *approximately stationary* at $\bar{x}$ if $\hat{\theta}[\mathbf{\Omega}](\bar{x}) = 0$
- *normally approximately stationary* at $\bar{x}$ if $\hat{\eta}[\mathbf{\Omega}](\bar{x}) = 0$

Definition

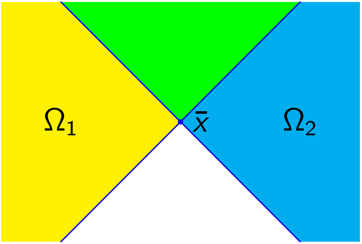
$\mathbf{\Omega}$ is

- *approximately stationary* at $\bar{x}$ if $\hat{\theta}[\mathbf{\Omega}](\bar{x}) = 0$
- *normally approximately stationary* at $\bar{x}$ if $\hat{\eta}[\mathbf{\Omega}](\bar{x}) = 0$

- *uniformly regular* at $\bar{x}$ if $\hat{\theta}[\mathbf{\Omega}](\bar{x}) > 0$
- *normally uniformly regular* at $\bar{x}$ if $\hat{\eta}[\mathbf{\Omega}](\bar{x}) > 0$

Stationarity and Regularity of Finite Collections

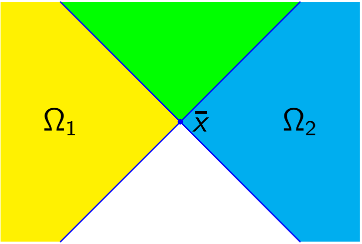
Regularity vs Stationarity



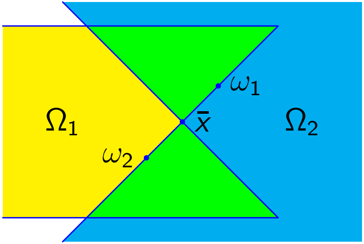
$$\hat{\theta}[\mathbf{\Omega}](\bar{x}) = \hat{\eta}[\mathbf{\Omega}](\bar{x}) > 0$$

Stationarity and Regularity of Finite Collections

Regularity vs Stationarity



$$\hat{\theta}[\mathbf{\Omega}](\bar{x}) = \hat{\eta}[\mathbf{\Omega}](\bar{x}) > 0$$



$$\hat{\theta}[\mathbf{\Omega}](\bar{x}) = \hat{\eta}[\mathbf{\Omega}](\bar{x}) = 0$$

Ω_i, $i \in I$, are closed

Theorem

- $\hat{\theta}[\mathbf{\Omega}](\bar{x}) \leq \hat{\eta}[\mathbf{\Omega}](\bar{x})$
- *If X is Asplund then* $\hat{\theta}[\mathbf{\Omega}](\bar{x}) = \hat{\eta}[\mathbf{\Omega}](\bar{x})$

Ω_i, $i \in I$, are closed

Theorem

- $\hat{\theta}[\mathbf{\Omega}](\bar{x}) \le \hat{\eta}[\mathbf{\Omega}](\bar{x})$
- *If X is Asplund then* $\hat{\theta}[\mathbf{\Omega}](\bar{x}) = \hat{\eta}[\mathbf{\Omega}](\bar{x})$

Extended Extremal Principle

$\mathbf{\Omega}$ is approximately stationary at $\bar{x}$ if and only if it is normally approximately stationary at $\bar{x}$

Ω_i, $i \in I$, are closed

Theorem

- $\hat{\theta}[\mathbf{\Omega}](\bar{x}) \le \hat{\eta}[\mathbf{\Omega}](\bar{x})$
- *If X is Asplund then* $\hat{\theta}[\mathbf{\Omega}](\bar{x}) = \hat{\eta}[\mathbf{\Omega}](\bar{x})$

Extended Extremal Principle

$\mathbf{\Omega}$ is approximately stationary at $\bar{x}$ if and only if it is normally approximately stationary at $\bar{x}$

Theorem

Extremal Principle holds $\Leftrightarrow$ *Extended Extremal Principle holds* $\Leftrightarrow$ *X is Asplund*

Stationarity and Regularity of Infinite Collections

X – Banach space,
$\boldsymbol{\Omega} := \{\Omega_i\}_{i\in I} \subset X$, $\quad |I| > 1$, $\quad \bar{x} \in \bigcap_{i\in I} \Omega_i$

Stationarity and Regularity of Infinite Collections

X – Banach space,
$\boldsymbol{\Omega} := \{\Omega_i\}_{i\in I} \subset X, \quad |I| > 1, \quad \bar{x} \in \bigcap_{i\in I} \Omega_i$
$\mathcal{J} := \{J \subset I |\ 1 < |J| < \infty\}$

Stationarity and Regularity of Infinite Collections

X – Banach space,
$\mathbf{\Omega} := \{\Omega_i\}_{i\in I} \subset X$, $\quad |I| > 1$, $\quad \bar{x} \in \bigcap_{i\in I} \Omega_i$
$\mathcal{J} := \{J \subset I \mid 1 < |J| < \infty\}$

$$\hat{\theta}[\mathbf{\Omega}](\bar{x}) := \sup_{\varepsilon>0} \inf_{\substack{\rho\in(0,\varepsilon),\ J\in\mathcal{J} \\ \omega_i\in B_\varepsilon(\bar{x})\cap\Omega_i\ (i\in J)}} \frac{\theta_\rho[\{\Omega_i - \omega_i\}_{i\in J}](0)}{\rho}$$

$$\hat{\eta}[\mathbf{\Omega}](\bar{x}) := \sup_{\varepsilon>0} \inf_{\substack{J\in\mathcal{J} \\ x_i\in\Omega_i\cap B_\varepsilon(\bar{x}),\, x_i^*\in N_{\Omega_i}(x_i)\ (i\in J), \\ \sum_{i\in J}\|x_i^*\|=1}} \left\|\sum_{i\in J} x_i^*\right\|$$

X – Banach space,
$\mathbf{\Omega} := \{\Omega_i\}_{i\in I} \subset X, \quad |I| > 1, \quad \bar{x} \in \bigcap_{i\in I} \Omega_i$
$\mathcal{J} := \{J \subset I |\ 1 < |J| < \infty\}$

Definition

$\mathbf{\Omega}$ is

- *approximately stationary* at $\bar{x}$ if $\hat{\theta}[\mathbf{\Omega}](\bar{x}) = 0$
- *normally approximately stationary* at $\bar{x}$ if $\hat{\eta}[\mathbf{\Omega}](\bar{x}) = 0$
- *uniformly regular* at $\bar{x}$ if $\hat{\theta}[\mathbf{\Omega}](\bar{x}) > 0$
- *normally uniformly regular* at $\bar{x}$ if $\hat{\eta}[\mathbf{\Omega}](\bar{x}) > 0$

X – Asplund space, Ω_i, $i \in I$, – closed, $\bar{x} \in \bigcap_{i\in I} \Omega_i$

Theorem

$\mathbf{\Omega}$ *is approximately stationary at* $\bar{x}$ *if and only if it is normally approximately stationary at* $\bar{x}$

X – Asplund space, Ω_i, $i \in I$, – closed, $\bar{x} \in \bigcap_{i\in I} \Omega_i$

Theorem

$\mathbf{\Omega}$ is approximately stationary at $\bar{x}$ if and only if it is normally approximately stationary at $\bar{x}$
Moreover, for any $\varepsilon > 0$, the corresponding properties are satisfied with the same set of indices J

Stationarity and Regularity of Infinite Collections

X – Asplund space, Ω_i, $i \in I$, – closed, $\bar{x} \in \bigcap_{i \in I} \Omega_i$

Theorem

$\boldsymbol{\Omega}$ *is approximately stationary at* $\bar{x}$ *if and only if it is normally approximately stationary at* $\bar{x}$
Moreover, for any $\varepsilon > 0$*, the corresponding properties are satisfied with the same set of indices* J

$\boldsymbol{\Omega}$ *is uniformly regular at* $\bar{x}$ *if and only if it is normally uniformly regular at* $\bar{x}$

X – Banach space,

$\mathbf{\Omega} := \{\Omega_i\}_{i\in I} \subset X, \quad |I| > 1, \quad \bar{x} \in \bigcap_{i\in I} \Omega_i$

$\mathcal{J} := \{J \subset I |\ 1 < |J| < \infty\}$

X – Banach space,
$\mathbf{\Omega} := \{\Omega_i\}_{i\in I} \subset X, \quad |I| > 1, \quad \bar{x} \in \bigcap_{i\in I} \Omega_i$
$\mathcal{J} := \{J \subset I \mid 1 < |J| < \infty\}$

$\Phi : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$
$\mathcal{J}_\alpha := \{J \subset I \mid 1 < |J| < \Phi(\alpha)\}$

X – Banach space,
$\mathbf{\Omega} := \{\Omega_i\}_{i\in I} \subset X, \quad |I| > 1, \quad \bar{x} \in \bigcap_{i\in I} \Omega_i$
$\mathcal{J} := \{J \subset I |\ 1 < |J| < \infty\}$

$\Phi : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$
$\mathcal{J}_\alpha := \{J \subset I |\ 1 < |J| < \Phi(\alpha)\}$

Definition

$\mathbf{\Omega}$ is *approximately Φ-stationary* at $\bar{x}$ if $\forall \varepsilon > 0\ \exists \rho \in (0, \varepsilon)$;
$\alpha \in (0, \varepsilon)$; $J \in \mathcal{J}_\alpha$; $\omega_i \in \Omega_i \cap B_\varepsilon(\bar{x})$, $a_i \in X$ $(i \in J)$

$$\max_{i\in J} \|a_i\| < \alpha\rho \quad \text{and} \quad \bigcap_{i\in J}(\Omega_i - \omega_i - a_i) \bigcap (\rho\mathbb{B}) = \emptyset$$

Stationarity and Regularity of Infinite Collections

Φ-stationarity vs Φ-regularity

X – Banach space,
$\mathbf{\Omega} := \{\Omega_i\}_{i\in I} \subset X, \quad |I| > 1, \quad \bar{x} \in \bigcap_{i\in I} \Omega_i$
$\mathcal{J} := \{J \subset I \mid 1 < |J| < \infty\}$

$\Phi : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$
$\mathcal{J}_\alpha := \{J \subset I \mid 1 < |J| < \Phi(\alpha)\}$

Definition

$\mathbf{\Omega}$ is *normally approximately Φ-stationary* at $\bar{x}$ if $\forall \varepsilon > 0\ \exists \alpha \in (0, \varepsilon)$;
$J \in \mathcal{J}_\alpha$; $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$, $x_i \in N_{\Omega_i}(x_i)$ $(i \in J)$

$$\left\| \sum_{i\in J} x_i^* \right\| < \varepsilon \sum_{i\in J} \|x_i^*\|$$

X – Asplund space, Ω_i, $i \in I$, – closed, $\bar{x} \in \bigcap_{i\in I} \Omega_i$

$\Phi : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$

Theorem

$\boldsymbol{\Omega}$ *is approximately* Φ*-stationary at* $\bar{x}$ *if and only if it is normally approximately* Φ*-stationary at* $\bar{x}$
Moreover, for any $\varepsilon > 0$*, the corresponding properties are satisfied with the same set of indices* J

$\boldsymbol{\Omega}$ *is uniformly* Φ*-regular at* $\bar{x}$ *if and only if it is normally uniformly* Φ*-regular at* $\bar{x}$

Intersection Rule

Fréchet Finite Normals

X – Asplund space, Ω_i, $i \in I$, – closed, $\bar{x} \in \bigcap_{i \in I} \Omega_i$

Intersection Rule

Fréchet Finite Normals

X – Asplund space, Ω_i, $i \in I$, – closed, $\bar{x} \in \bigcap_{i \in I} \Omega_i$
$\mathcal{J} := \{J \subset I \mid 0 < |J| < \infty\}$

Intersection Rule

Fréchet Finite Normals

X – Asplund space, Ω_i, $i \in I$, – closed, $\bar{x} \in \bigcap_{i\in I} \Omega_i$
$\mathcal{J} := \{J \subset I |\ 0 < |J| < \infty\}$

Definition

$x^* \in X^*$ is *Fréchet finitely normal* to $\bigcap_{i\in I} \Omega_i$ at $\bar{x}$ if $\forall \varepsilon > 0\ \exists \rho > 0$ and $J \in \mathcal{J}$

$$\langle x^*, x - \bar{x} \rangle < \varepsilon \|x - \bar{x}\| \quad \forall x \in \bigcap_{i\in J} \Omega_i \bigcap B_\rho(\bar{x}) \setminus \{\bar{x}\}$$

Intersection Rule

X – Asplund space, Ω_i, $i \in I$, – closed, $\bar{x} \in \bigcap_{i\in I} \Omega_i$
$\mathcal{J} := \{J \subset I |\ 0 < |J| < \infty\}$

Theorem

If $x^* \in X^*$ is Fréchet finitely normal to $\bigcap_{i\in I} \Omega_i$ at $\bar{x}$, then $\forall \varepsilon > 0$ $\exists J \in \mathcal{J}$; $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$, $x_i^* \in N_{\Omega_i}(x_i)$ $(i \in J)$; $\lambda \geq 0$

$$\sum_{i\in J} \|x_i^*\| + \lambda = 1 \quad \textit{and} \quad \left\| \lambda x^* - \sum_{i\in J} x_i^* \right\| < \varepsilon$$

X – Asplund space, Ω_i, $i \in I$, – closed, $\bar{x} \in \bigcap_{i\in I} \Omega_i$
$\mathcal{J} := \{J \subset I |\ 0 < |J| < \infty\}$

Corollary

Suppose $\mathbf{\Omega}$ *is Fréchet normally uniformly regular at* $\bar{x}$*. If* $x^* \in X^*$ *is finitely normal to the intersection* $\bigcap_{i\in I} \Omega_i$ *at* $\bar{x}$*, then* $\forall \varepsilon > 0\ \exists J \in \mathcal{J}$*;* $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$*,* $x_i^* \in N_{\Omega_i}(x_i)$ $(i \in J)$

$$\left\| x^* - \sum_{i\in J} x_i^* \right\| < \varepsilon$$

Constrained Optimization
Finite Stationarity

$$\text{Minimize} \quad f_0(x) \quad \text{subject to} \quad f_i(x) \le 0,\ i \in I$$

$$f_i : X \to \mathbb{R}_\infty,\ f_i(\bar{x}) < \infty\ (i \in I \cup \{0\}), \qquad f_0(\bar{x}) = 0$$

Constrained Optimization

Finite Stationarity

$$\text{Minimize} \quad f_0(x) \quad \text{subject to} \quad f_i(x) \le 0,\ i \in I$$

$$f_i : X \to \mathbb{R}_\infty,\ f_i(\bar{x}) < \infty \ (i \in I \cup \{0\}), \qquad f_0(\bar{x}) = 0$$

Definition

$\bar{x}$ is *finitely stationary* if $\forall \varepsilon > 0\ \exists \rho > 0$ and $J \in \mathcal{J}$

$$\sup_{i \in J \cup \{0\}} f_i(x) + \varepsilon \|x - \bar{x}\| > 0 \quad \forall x \in B_\rho(\bar{x}) \setminus \{\bar{x}\}$$

Constrained Optimization

Finite Stationarity

$$\text{Minimize} \quad f_0(x) \quad \text{subject to} \quad f_i(x) \le 0,\ i \in I$$

$$f_i : X \to \mathbb{R}_\infty,\ f_i(\bar{x}) < \infty\ (i \in I \cup \{0\}), \qquad f_0(\bar{x}) = 0$$

Definition

$\bar{x}$ is *finitely stationary* if $\forall \varepsilon > 0\ \exists \rho > 0$ and $J \in \mathcal{J}$

$$\sup_{i \in J \cup \{0\}} f_i(x) + \varepsilon \|x - \bar{x}\| > 0 \quad \forall x \in B_\rho(\bar{x}) \setminus \{\bar{x}\}$$

ε*-active* indices: $I_\varepsilon(\bar{x}) := \left\{ i \in I \;\middle|\; \sup_{x \in B_\varepsilon(\bar{x})} f_i(x) \ge -\varepsilon \right\}$

Constrained Optimization

Finite Stationarity

$$\text{Minimize} \quad f_0(x) \quad \text{subject to} \quad f_i(x) \le 0,\ i \in I$$

$$f_i : X \to \mathbb{R}_\infty,\ f_i(\bar{x}) < \infty\ (i \in I \cup \{0\}), \qquad f_0(\bar{x}) = 0$$

Definition

$\bar{x}$ is *finitely stationary* if $\forall \varepsilon > 0\ \exists \rho > 0$ and $J \in \mathcal{J}$

$$\sup_{i \in J \cup \{0\}} f_i(x) + \varepsilon \|x - \bar{x}\| > 0 \quad \forall x \in B_\rho(\bar{x}) \setminus \{\bar{x}\}$$

ε-active indices: $I_\varepsilon(\bar{x}) := \left\{ i \in I \;\middle|\; \sup\limits_{x \in B_\varepsilon(\bar{x})} f_i(x) \ge -\varepsilon \right\}$

$$\mathcal{J}_\varepsilon(\bar{x}) := \{J \subset I_\varepsilon(\bar{x}) |\ 0 < |J| < \infty\}$$

$$\text{Minimize} \quad f_0(x) \quad \text{subject to} \quad f_i(x) \le 0,\ i \in I$$

$f_i : X \to \mathbb{R}_\infty$, $f_i(\bar{x}) < \infty$ $(i \in I \cup \{0\})$, $\quad f_0(\bar{x}) = 0$

Definition

$\{f_i\}_{i \in I \cup \{0\}}$ is *normally uniformly regular* at $\bar{x}$ if $\exists \alpha > 0$, $\varepsilon > 0$

$$\left\| \sum_{i \in J \cup \{0\}} x_i^* \right\| + \sum_{i \in J \cup \{0\}} \lambda_i \ge \alpha \sum_{i \in J \cup \{0\}} \|x_i^*\|$$

$\forall J \in \mathcal{J}_\varepsilon(\bar{x})$;
$\forall (x_i, \mu_i) \in \mathrm{epi}\, f_i \cap B_\varepsilon(\bar{x}, 0)$, $(x_i^*, -\lambda_i) \in N_{\mathrm{epi}\, f_i}(x_i, \mu_i)$ $(i \in J \cup \{0\})$

Constrained Optimization

Finite Stationarity

$$\text{Minimize} \quad f_0(x) \quad \text{subject to} \quad f_i(x) \le 0,\ i \in I$$

$$f_i : X \to \mathbb{R}_\infty,\ f_i(\bar{x}) < \infty\ (i \in I \cup \{0\}), \qquad f_0(\bar{x}) = 0$$

Theorem

Suppose f_i, $i \in I \cup \{0\}$, are lsc near $\bar{x}$ and $\{f_i\}_{i \in I \cup \{0\}}$ is normally uniformly regular at $\bar{x}$. If $\bar{x}$ is finitely stationary, then $\forall \varepsilon > 0$, $\exists J \in \mathcal{J}_\varepsilon(\bar{x})$; $x_i \in B_\varepsilon(\bar{x})$, $x_i^* \in X^*$, $\lambda_i \ge 0$ $(i \in J \cup \{0\})$

$$f_i(x_i) \le f(\bar{x}) + \varepsilon;\ x_i^* \in \lambda_i \partial f_i(x_i) \text{ if } \lambda_i > 0,\ x_i^* \in \partial^\infty f_i(x_i) \text{ if } \lambda_i = 0$$

$$\left\| \sum_{i \in J \cup \{0\}} x_i^* \right\| < \varepsilon \quad \text{and} \quad \sum_{i \in J \cup \{0\}} \lambda_i = 1$$

Constrained Optimization

Finite Stationarity

$$\text{Minimize} \quad f_0(x) \quad \text{subject to} \quad f_i(x) \le 0,\ i \in I$$

$$f_i : X \to \mathbb{R}_\infty,\ f_i(\bar{x}) < \infty\ (i \in I \cup \{0\}), \qquad f_0(\bar{x}) = 0$$

Corollary

Suppose f_i, $i \in I \cup \{0\}$, are uniformly Lipschitz near $\bar{x}$, and $\{f_i\}_{i \in I \cup \{0\}}$ is normally uniformly regular at $\bar{x}$. If $\bar{x}$ is finitely stationary, then $\forall \varepsilon > 0$, $\exists J \in \mathcal{J}_\varepsilon(\bar{x})$; $x_i \in B_\varepsilon(\bar{x})$, $x_i^* \in \partial f_i(x_i)$, $\lambda_i \ge 0$ $(i \in J \cup \{0\})$

$$\left\| \sum_{i \in J \cup \{0\}} \lambda_i x_i^* \right\| < \varepsilon \quad \textit{and} \quad \sum_{i \in J \cup \{0\}} \lambda_i = 1$$

Constrained Optimization
Finite Stationarity

$$\text{Minimize} \quad f_0(x) \quad \text{subject to} \quad f_i(x) \le 0,\ i \in I$$

$$f_i : X \to \mathbb{R}_\infty,\ f_i(\bar{x}) < \infty\ (i \in I \cup \{0\}), \qquad f_0(\bar{x}) = 0$$

Definition

Normal constraint qualification: $\exists \alpha > 0,\ \varepsilon > 0$

$$\left\| \sum_{i \in J} x_i^* \right\| \ge \alpha \sum_{i \in J} \lambda_i$$

$\forall J \in \mathcal{J}_\varepsilon(\bar{x})$; $(x_i, \mu_i) \in \operatorname{epi} f_i \cap B_\varepsilon(\bar{x}, 0)$, $(x_i^*, -\lambda_i) \in N_{\operatorname{epi} f_i}(x_i, \mu_i)$ $(i \in J)$

Constrained Optimization

Finite Stationarity

$$\text{Minimize} \quad f_0(x) \quad \text{subject to} \quad f_i(x) \le 0,\ i \in I$$

$$f_i : X \to \mathbb{R}_\infty,\ f_i(\bar{x}) < \infty\ (i \in I \cup \{0\}), \qquad f_0(\bar{x}) = 0$$

Corollary

Suppose f_i, $i \in I \cup \{0\}$, are uniformly Lipschitz near $\bar{x}$, $\{f_i\}_{i \in I \cup \{0\}}$ is normally uniformly regular at $\bar{x}$, and the normal constraint qualification is satisfied. If $\bar{x}$ is finitely stationary, then $\forall \varepsilon > 0$ $\exists J \in \mathcal{J}_\varepsilon(\bar{x})$; $x_i \in B_\varepsilon(\bar{x})$, $x_i^* \in \partial f_i(x_i)$ $(i \in J \cup \{0\})$; $\lambda_i \ge 0$ $(i \in J)$

$$\left\| x_0^* + \sum_{i \in J} \lambda_i x_i^* \right\| < \varepsilon$$

References

1. A. Y. Kruger and B. S. Mordukhovich, *Extremal points and the Euler equation in nonsmooth optimization*, Dokl. Akad. Nauk BSSR **24**:8 (1980), 684–687, in Russian.
2. B. S. Mordukhovich and Y. Shao, *Extremal characterizations of Asplund spaces*, Proc. Amer. Math. Soc. **124** (1996), 197–205.
3. A. Y. Kruger, *Weak stationarity: eliminating the gap between necessary and sufficient conditions*, Optimization **53** (2004), 147–164.
4. A. Y. Kruger, *Stationarity and regularity of set systems*, Pacif. J. Optimiz. **1** (2005), 101–126.
5. A. Y. Kruger, *About regularity of collections of sets*, Set-Valued Anal. **14** (2006), 187–206.
6. A. Y. Kruger, *About stationarity and regularity in variational analysis*, Taiwanese J. Math. **13** (2009), 1737–1785.

References

1. A. Y. Kruger and M. A. López, *Stationarity and regularity of infinite collections of sets*, J. Optim. Theory Appl. **154** (2012).
2. A. Y. Kruger and M. A. López, *Stationarity and regularity of infinite collections of sets. Applications to infinitely constrained optimization*, J. Optim. Theory Appl. **155** (2012).


Thank you