Accuracy and Robustness under Game Theoretic Framework

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Outline

1. Introduction
   - Reasons for Uncertainty
   - Sensitivity and Robustness

2. Boolean Program under Game Theory Formulation
   - Problem Formulation
   - Deviation Measures

3. Main Results
   - Accuracy Functions
   - Accuracy Radius

4. Example

5. Conclusion
Reasons for Uncertainty

Uncertainty in **Optimization**
- inaccuracy of initial data
- non-adequacy of models to real processes
- errors of numerical methods
- errors of rounding off
- absence or lack of precise and reliable information etc.

Uncertainty in **Scheduling and Time-tabling**
- equipment (machine) breakdowns or activity (job) disruption
- earliness or tardiness, changes of job processing
- changes of release dates, due dates, deadlines or resource availability etc.
Discrete Optimization Models with Uncertainty

Post-optimal and parametric analysis

- Sensitivity and tolerances:
  - Libura 2000;

- Stability analysis in vector discrete models:
  - Emelichev et al. 1995;

- Stability analysis in scheduling theory:
  - Sotskov and Werner 1998;

Robust optimization, worst-case analysis

- Minmax Regret Optimization:
  - Kouvelis and Yu 1997;

- Flexible Robust Optimization:
  - Bertsimas and Sim 2003;

- Absolute and Relative Robustness:
  - Yaman et al. 2001,
  - Montemanni and Gambardella 2002,
Let

- \( m \geq 2 \) be the number of players;
- \( N_m := \{1, 2, \ldots, m\} \) be a set of players;
- \( X_i, |X_i| := \{0, 1\} \) be a finite set of strategies of player \( i \);
- Boolean variables \( x_i \) encode the choice of strategy for player \( i \);

\[
X \subset \prod_{i \in N_m} X_i = \{0, 1\}^m, \quad 0_{(m)} := (0, 0, \ldots, 0)^T \notin X.
\]

be the set of feasible game situations (solutions)
A vector of payoff functions (payoff profile)

\[ p(C, x) := (p_1(C, x), \ldots, p_m(C, x))^T, \]

\[ p_i(C, x) := C_i x, \quad i \in N_m \]

Here \( C_i \) is \( i \)-th row of matrix \( C = [c_{ij}] \in \mathbb{R}^{m \times m}, \quad x := (x_1, x_2, \ldots, x_m)^T, \quad x_i \in X_i. \)

Thus, a set of payoff profiles is

\[ PP(C, X) := \{p(C, x) : x \in X\}. \]
Pareto and Nash equilibria

Pareto equilibrium

A solution $x^* \in X$ is called **Pareto equilibrium** in the strategic game with matrix $C$ if there are no solution $x \in X$ such that $p_i(C, x) \leq p_i(C, x^*)$ for all $i \in N_m$, and $p_j(C, x) < p_j(C, x^*)$ for some $j \in N_m$. 

$$W_i(x^*) := X_i \times \prod_{j \in N_m \setminus \{i\}} x_j^* = \{(x_1^*, x_2^*, \ldots, x_i^*, \ldots, x^*_m), (x_1^*, x_2^*, \ldots, \bar{x}_i^*, \ldots, x^*_m)\}.$$ 

Nash equilibrium

A solution $x^* \in X$ is called **Nash equilibrium** in the strategic game with matrix $C$ if for every player $i \in N_m$ the following inequality holds $p_i(C, x^*) \leq p_i(C, x)$ for all $x \in W_i(x^*) \cap X$.

Denote $P^m(C)$ and $N^m(C)$ the set of Pareto and Nash equilibria, respectively.
Relative deviation and error

Relative deviation

For \( x, \tilde{x} \in X \) and \( C \in \mathbb{R}^{m \times m}_+ \), denote

\[
\Delta_i(C, \tilde{x}, x) := \frac{p_i(C, \tilde{x}) - p_i(C, x)}{p_i(C, x)}, \ i \in N_m.
\]

Definition

For \( x^* \in X \) and \( C \in \mathbb{R}^{m \times m}_+ \), the relative error is defined as follows:

in the Pareto case

\[
\epsilon_P(C, x^*) := \max_{x \in X} \min_{i \in N_m} \Delta_i(C, x^*, x).
\]

in the Nash case

\[
\epsilon_N(C, x^*) := \max_{i \in N_m} \max_{x \in W_i(x^*) \cap X} \Delta_i(C, x^*, x).
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\]
Relative error as a measure of solution quality with respect to optimality

\[ x^* \in N^m(C) \text{ iff } \epsilon_N(C, x^*) = 0. \]
\[ x^* \in P^m(C) \text{ iff } \epsilon_N(C, x^*) = 0 \text{ not always true!} \]

**Example**

Let \( m = 2 \) and \( \tilde{C} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \). Assume that \( X = \{x^1, x^2\} \), \( x^1 = (0, 1)^T \), \( x^2 = (1, 0)^T \). Then \( PP(\tilde{C}, X) = \{(2, 1)^T, (1, 2)^T\} \). If we consider the matrix \( \tilde{C} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \), then \( PP(\tilde{C}, X) = \{(1, 1)^T, (1, 2)^T\} \). Evidently, \( x^2 \in P^2(\tilde{C}) \) and \( \epsilon_P(\tilde{C}, x^2) = 0 \), but \( x^2 \notin P^2(\tilde{C}) \) and \( \epsilon_P(\tilde{C}, x^2) = 0 \).

However \( x^* \in P^m(C) \text{ iff } \exists \phi > 0 \epsilon_P(C', x^*) = 0 \forall C', \|C' - C\| < \phi \).
Relative error as a measure of solution quality with respect to optimality

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However \( x^* \in P^m(C) \) iff \( \exists \phi > 0 \varepsilon_P(C', x^*) = 0 \forall C', ||C' - C|| < \phi. \)
For a given $\delta \in [0, 1)$, we consider a set of perturbed matrices
\[ \Theta_\delta(C^0) := \{ C \in \mathbb{R}^{m \times m}_+ : |c_{ij} - c_{ij}^0| \leq \delta \cdot c_{ij}^0, \ i \in N_m, \ j \in N_m \}. \]

**Definition**

For $x^* \in X$ and $\delta \in [0, 1)$, the value of the accuracy function is defined as follows:

- **in the Pareto case**
  \[ A_P(C^0, x^*, \delta) := \max_{C \in \Theta_\delta(C^0)} \varepsilon_P(C, x^*); \]

- **in the Nash case**
  \[ A_N(C^0, x^*, \delta) := \max_{C \in \Theta_\delta(C^0)} \varepsilon_N(C, x^*). \]
In single objective case (one player $i$ makes a decision only), we have for $x^* \in X$ and $\delta \in [0, 1)$, the value of the accuracy function is defined as follows:

$$A_i(C^0, x^*, \delta) := \max_{C_i \in \Theta_\delta(C^0_i)} \varepsilon_i(C, x^*) = \max_{C_i \in \Theta_\delta(C^0_i)} \left( \frac{p_i(C, \tilde{x}) - \min_{x \in X} p_i(C, x)}{\min_{x \in X} p_i(C, x)} \right),$$

which is similar to the robust deviation measure (if absolute deviation is used)
Theorem

(i) For \(x^* \in X\) and \(\delta \in [0, 1)\), the accuracy function can be expressed by the formula:

\[
A_P(C^0, x^*, \delta) = \max_{x \in X} \min_{i \in N_m} \frac{C_i^0 (x^* - x) + \delta \sum_{j \in N_m} c_{ij}^0 |x_j^* - x_j|}{(1 - \delta)C_i^0 x}.
\] (1)

(ii) For \(x^* \in X\) and \(\delta \in [0, 1)\), the accuracy function can be expressed by the formula:

\[
A_N(C^0, x^*, \delta) = \max_{i \in N_m} \max_{x \in W_i(x^*) \cap X} \frac{C_i^0 (x^* - x) + \delta \sum_{j \in N_m} c_{ij}^0 |x_j^* - x_j|}{(1 - \delta)C_i^0 x}.
\] (2)
Efficient Bounds for Accuracy Functions

**Proposition**

For $x^* \in X$ and $\delta \in [0, 1)$,

$$A_P(C^0, x^*, \delta) \leq \frac{2\delta}{1-\delta} + \frac{1+\delta}{1-\delta} \cdot \min_{i \in N_m} A_i(C^0, x^*, 0).$$  \(3\)

**Corollary**

For $x^* \in N^m(C^0)$ and $\delta \in [0, 1)$, the equality $A_N(C^0, x^*, \delta) = 0$ holds.
For $x^* \in X$, the value of the accuracy function is defined as follows:

$$R_{P,N}(C^0, x^*) := \sup \left\{ \delta \in [0, 1) : A_{P,N}(C^0, x^*, \delta) = 0 \right\}.$$ 

**Theorem**

(i) For $x^* \in P^m(C^0)$, the accuracy radius can be expressed by the formula:

$$R_{P}(C^0, x^*) = \min \left\{ 1, \min_{x \in X \setminus \{x^*\}} \max_{i \in N_m} \frac{C^0_i(x - x^*)}{\sum_{j \in N_m} c^0_{ij} |x_j - x^*_j|} \right\}. \quad (4)$$

(ii) For $x^* \in N^m(C^0)$, the accuracy radius can be expressed by the formula:

$$R_{N}(C^0, x^*) = \min \left\{ 1, \min_{i \in N_m} \min_{x \in W_i(x^*) \cap X \setminus \{x^*\}} \frac{C^0_i(x - x^*)}{\sum_{j \in N_m} c^0_{ij} |x_j - x^*_j|} \right\} = 1, \quad (5)$$

i.e. $x^* \in N^m(C^0)$ is accurate.
Example

Let $m = 4$, and

\[ C^0 = \begin{pmatrix} 2 & 2 & 1.5 & 2 \\ 0.5 & 1 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 3 & 3 & 3 \end{pmatrix}. \]

Let $X = \{ x^1, x^2, x^3, x^4 \}$, $x^1 = (0, 1, 1, 0)^T$, $x^2 = (1, 0, 1, 0)^T$, $x^3 = (0, 1, 0, 1)^T$, $x^4 = (0, 0, 1, 1)^T$.

Then $C^0 x^1 = (3.5, 3, 2, 6)^T$, $C^0 x^2 = (3.5, 2.5, 3, 4)^T$, $C^0 x^3 = (4, 2, 3, 6)^T$, $C^0 x^4 = (3.5, 3, 3, 6)^T$ and $P^3(C^0) = \{ x^1, x^2, x^3 \}$.

We calculate

\[ R^4(C^0, x^1) = \frac{1}{3}, \quad R^4(C^0, x^2) = \frac{1}{5}, \quad R^4(C^0, x^3) = \frac{1}{9}. \]
Example: accuracy functions and radii

Figure: Accuracy functions; \( \rho \in [0, 1) \).
Example: accuracy functions and radii
Similarities and Differences:

- Basic tools applied in robust and accuracy analysis are very much similar
- Ranking alternatives based on calculating accuracy measures vs. robust optimization

Challenges and Obstacles:

- Lack of efficient algorithms: heuristic approaches should be helpful!
- Non-linear models are difficult to be analyzed
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Most Important References


Thank You For Your Time and Interest!

Questions?
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