Generalizing Trade-off Directions in Multiobjective Optimization

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Outline

1 Introduction
   - Problem Formulation
   - Main Definitions
   - Main Tools

2 Main Results
   - Convex Case
   - Nonconvex Case

3 Concluding Remarks
Optimality under Multiple Objectives

- Different concepts of optimality in multiobjective optimization
- Various definitions of trade-off
We consider general multiobjective optimization problems of the following form:

\[
\min_{x \in S} \{ f_1(x), f_2(x), \ldots, f_k(x) \},
\]

with the continuous objective functions \( f_i : \mathbb{R}^n \rightarrow \mathbb{R} \) for all \( i \in I_k := \{1, \ldots, k\} \).

The decision vector \( x \) belongs to the nonempty feasible set \( S \subset \mathbb{R}^n \).

Elements of the image of the feasible set \( Z \subset \mathbb{R}^k \) are termed objective vectors and denoted by

\[
z = f(x) = (f_1(x), f_2(x), \ldots, f_k(x))^T.
\]

Additionally, we assume \( f(B(x; \varepsilon)) \) to be open for all \( x \in S \) and \( \varepsilon > 0 \), where \( B(x; \varepsilon) \) is an open ball with radius \( \varepsilon \) and center \( x \).
Efficiency and Weak Pareto Optimality

**Weak Pareto Optimality.** An objective vector \( z^* \in Z \) is *weakly Pareto optimal* if there does not exist another objective vector \( z \in Z \) such that \( z_i < z_i^* \) for all \( i \in I_k \).

**Pareto Optimality or Efficiency.** An objective vector \( z^* \in Z \) is *Pareto optimal or efficient* if there does not exist another objective vector \( z \in Z \) such that \( z_i \leq z_i^* \) for all \( i \in I_k \) and \( z_j < z_j^* \) for at least one index \( j \).
The \textit{weakly Pareto optimal set} is

\[ GW P(Z) := \left\{ z \in Z \mid (z + \text{int } \mathbb{R}^k_{-}) \cap Z = \emptyset \right\}; \]

the \textit{Pareto optimal set} is

\[ GPO(Z) := \left\{ z \in Z \mid (z + \mathbb{R}^k_{-} \setminus \{0\}) \cap Z = \emptyset \right\}. \]
Local Optimality

**Definition**

The *locally weakly Pareto optimal set* with \( z = f(x) \in Z \) is given as

\[
LWP(Z) = \bigcup_{\delta > 0} \left\{ z \in Z \mid (z + \text{int} \ R_+^k) \cap Z \cap f(B(x; \delta)) = \emptyset \right\}.
\]

The *locally Pareto optimal set* as

\[
LPO(Z) = \bigcup_{\delta > 0} \left\{ z \in Z \mid (z + R_+^k \setminus \{0\}) \cap Z \cap f(B(x; \delta)) = \emptyset \right\}.
\]
**Strong Efficiency.** An objective vector $z^* \in Z$ is *strongly Pareto optimal* if for all $i \in I_k$ there exists no objective vector $z \in Z$ such that $z_i < z_i^*$ or in other words $z^* \in Z$ optimizes all objectives $z_i$, $i \in I_k$.

**Proper Pareto Optimality.** An objective vector $z^* \in Z$ is *properly Pareto optimal* if there exists no objective vector $z \in Z$ such that $z \in C$ for some convex cone $C$, $\mathbb{R}_+^k \setminus \{0\} \subset \text{int } C$, attached to $z^*$. 

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Generalizing Trade-off Directions in Multiobjective Optimization
Strong Efficiency and Proper Pareto Optimality via Cones

Definition

The *strongly efficient set* is

\[ GSE(Z) := \left\{ z \in Z \mid (z + (R^k_+)^C) \cap Z = \emptyset \right\}. \]

The *properly Pareto optimal set* is defined as

\[ GPP(Z) := \left\{ z \in Z \mid (z + C \setminus \{0\}) \cap Z = \emptyset \right\} \]

for some convex cone \( C \) such that \( R^k_- \setminus \{0\} \subset \text{int} \, C. \)
Local Optimality

Definition

The *locally strongly efficient set* with $z = f(x)$ is defined as

$$LSE(Z) := \bigcup_{\delta > 0} \left\{ z \in Z \mid (z + (\mathbb{R}_+^k)^C) \cap Z \cap f(B(x; \delta)) = \emptyset \right\}.$$ 

The *locally properly Pareto optimal set* as

$$LPP(Z) = \bigcup_{\delta > 0} \left\{ z \in Z \mid (z + C \setminus \{0\}) \cap Z \cap f(B(x; \delta)) = \emptyset \right\}$$

for some convex cone $C$ such that $\mathbb{R}_-^k \setminus \{0\} \subset \text{int} \ C$. 

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- Main Results
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Generalizing Trade-off Directions in Multiobjective Optimization
Contingent and Normal Cones

Definition

The **contingent cone** of a set $Z \subset \mathbb{R}^k$ at $z \in Z$ is defined as

$$K_z(Z) := \{ d \in \mathbb{R}^k \mid \text{there exist } t_j \searrow 0 \text{ and } d_j \to d \text{ such that } z + t_j d_j \in Z \}.$$

Definition

The **normal cone** of $Z$ at $z \in Z$ is the polar cone of the contingent cone, that is,

$$N_z(Z) := K_z(Z)^\circ = \{ y \in \mathbb{R}^k \mid y^T d \leq 0 \text{ for all } d \in K_z(Z) \}.$$
Contingent and Normal Cones

Definition

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Contingent cone may lose convexity in nonconvex case

Figure: Nonconvex contingent cone $K_z(Z)$. 
The cone of globally feasible directions of a set $Z \subset \mathbb{R}^k$ at $z \in Z$ is denoted by

$$D_z(Z) := \{d \in \mathbb{R}^k | \text{ there exists } t > 0 \text{ such that } z + td \in Z\}.$$ 

The set $Z$ is called regular at $z \in Z$ if $D_z(Z) = K_z(Z)$. 
Definition

The cone of globally feasible directions of a set $Z \subset \mathbb{R}^k$ at $z \in Z$ is denoted by

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The set $Z$ is called regular at $z \in Z$ if $D_z(Z) = K_z(Z)$. 
The cone of locally feasible directions of a set \( Z \subset \mathbb{R}^k \) at \( z \in Z \) is denoted by

\[
F_z(Z) = \{ d \in \mathbb{R}^k \mid \text{there exists } t > 0 \text{ such that } z + \tau d \in Z \text{ for all } \tau \in (0, t]\}.
\]

The set \( Z \) is called locally regular at \( z \in Z \) if \( F_z(Z) = K_z(Z) \).
The cone of locally feasible directions of a set $Z \subset \mathbb{R}^k$ at $z \in Z$ is denoted by

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The set $Z$ is called locally regular at $z \in Z$ if $F_z(Z) = K_z(Z)$. 

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The **tangent cone** of a set \( Z \subset \mathbb{R}^k \) at \( z \in Z \) is given by the formula

\[
T_z(Z) = \{ d \in \mathbb{R}^k \mid \text{for all } t_j \downarrow 0 \text{ and } z_j \to z \text{ with } z_j \in Z, \quad \text{there exists } d_j \to d \text{ with } z_j + t_j d_j \in Z \}.
\]

The set \( Z \) is called **tangentially regular** at \( z \in Z \) if \( T_z(Z) = K_z(Z) \).
Tangent Cone and Tangent Regularity

**Definition**

The *tangent cone* of a set $Z \subset \mathbb{R}^k$ at $z \in Z$ is given by the formula

$$T_z(Z) = \{ d \in \mathbb{R}^k \mid$$

for all $t_j \searrow 0$ and $z_j \to z$ with $z_j \in Z$,

there exists $d_j \to d$ with $z_j + t_j d_j \in Z \}.$

**Definition**

The set $Z$ is called *tangentially regular* at $z \in Z$ if $T_z(Z) = K_z(Z)$.
Lemma

For the cones $K_z(Z)$, $D_z(Z)$, $T_z(Z)$ and $F_z(Z)$ we have the following

a) $K_z(Z)$ and $T_z(Z)$ are closed and $T_z(Z)$ is convex.

b) $0 \in K_z(Z) \cap D_z(Z) \cap T_z(Z) \cap F_z(Z)$.

c) $Z \subset z + D_z(Z)$.

d) $\text{cl } F_z(Z) \subset K_z(Z) \subset \text{cl } D_z(Z)$.

e) $T_z(Z) \subset K_z(Z)$.

f) If $Z$ is convex, then $\text{cl } F_z(Z) = T_z(Z) = K_z(Z) = \text{cl } D_z(Z)$. Moreover $F_z(Z) = D_z(Z)$.
### Interconnection between various types of regularity

\[
\text{cl } D_z(Z) = K_z(Z) \quad \Leftarrow \quad \text{Regularity}
\]

\[\uparrow\]

\[\text{Convexity} \quad \Rightarrow \quad \text{Tangent Regularity}\]

\[\downarrow\]

\[
\text{cl } F_z(Z) = K_z(Z) \quad \Leftarrow \quad \text{Local Regularity}
\]

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The set of generalized *trade-off directions* is defined as:
- in case of *weak Pareto optimality*:
  \[ GW(Z) := GW \cdot P(K_Z(Z)) ; \]
- in case of *Pareto optimality (efficiency)*:
  \[ GP(Z) := G \cdot P(O)(K_Z(Z)) ; \]
- in case of *strong efficiency*:
  \[ GS(Z) := G \cdot S(E)(K_Z(Z)) . \]
Collection of the relationships in convex case with proper Pareto optimality

\[ D_z(Z) \cap \mathbb{R}_+^k = D_z(Z) \]
\[ z \in GSE(Z) \ \Leftrightarrow \ K_z(Z) \cap \mathbb{R}_+^k = K_z(Z) \]
\[ z \in GPP(Z) \ \Leftrightarrow \ K_z(Z) \cap \mathbb{R}_-^k \setminus \{0\} = \emptyset \]
\[ z \in GPO(Z) \ \Leftrightarrow \ D_z(Z) \cap \mathbb{R}_-^k \setminus \{0\} = \emptyset \]
\[ z \in GWP(Z) \ \Leftrightarrow \ D_z(Z) \cap \text{int} \ \mathbb{R}_-^k = \emptyset \]

* - tangent regularity, ** - local regularity, *** - regularity
Collection of the relationships in nonconvex case with proper Pareto optimality

\[ z \in LSE(Z) \implies K_z(Z) \cap \mathbb{R}_+^k = K_z(Z) \]
\[ \downarrow \]
\[ z \in LPP(Z) \iff K_z(Z) \cap \mathbb{R}_-^k \setminus \{0\} = \emptyset \]
\[ \Downarrow^{**} \]
\[ z \in LPO(Z) \iff F_z(Z) \cap \mathbb{R}_-^k \setminus \{0\} = \emptyset \]
\[ \downarrow \]
\[ z \in LW P(Z) \implies K_z(Z) \cap \text{int } \mathbb{R}_-^k = \emptyset \]

\[ * \iff N_z(Z) \cap \mathbb{R}_-^k = \mathbb{R}_-^k \]
\[ \Downarrow_* \iff G_S(Z) = \{0\} \]
\[ * \iff N_z(Z) \cap \text{int } \mathbb{R}_-^k \neq \emptyset \]
\[ \Downarrow_* \iff G_P(Z) \neq \emptyset \]
\[ ** \iff N_z(Z) \cap \text{int } \mathbb{R}_-^k \neq \emptyset \]
\[ \Downarrow_* \iff G_P(Z) \neq \emptyset \]
\[ ** \iff N_z(Z) \cap \mathbb{R}_-^k \setminus \{0\} \neq \emptyset \]
\[ \Downarrow_* \iff G_W(Z) \neq \emptyset \]

* - tangent regularity, ** - local regularity, *** - regularity
Characterization

**Theorem**

If $Z = z + C$, where $C$ is a closed cone, then

$$GPP(Z) = LPP(Z) = LPO(Z) = GPO(Z).$$
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Open Problems and Future Research

- To study problems with some nice properties close to convexity (e.g. with invex objective functions)
- To consider various generalized optimality principles which are given by means of either some ordering cone or parameterization
- How to exploit trade-off information in interactive methods?


Thank you for your time and interest!